

# ABSENCE OF SHOCKS FOR 1D EULER-POISSON SYSTEM

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## Abstract

It is shown that smooth solutions with small amplitude to the 1D Euler-Poisson system for electrons persist forever with no shock formation.

## 1 Introduction

In this paper, we consider the 1D Euler-Poisson system in plasma physics:

$$\begin{aligned} n_t + (nv)_x &= 0, \\ v_t + vv_x + \frac{1}{m_e n} p(n)_x &= \frac{e}{m_e} \psi_x \end{aligned} \tag{1.1}$$

with the electric field  $\psi_x$  which satisfies the Poisson equation

$$\psi_{xx} = 4\pi e(n - n_0), \text{ with } |\psi| \rightarrow 0, \text{ when } x \rightarrow \infty.$$

Here, the electrons of charge  $e$  and mass  $m_e$  are described by a density  $n(t, x)$  and an average velocity  $v(t, x)$ . The constant equilibrium-charged density of ions and electrons is  $\pm en_0$ .  $p$  denotes the pressure.

Euler-Poisson system (1.1) describes the simplest two-fluid model in plasma physics. In this model, the ions are treated as immobile and only form a constant charged background  $n_0$ . The two-fluid models describe dynamics of two separate compressible fluids of ions and electrons interacting with their self-consistent electromagnetic field. As pointed in the classical book of Jackson [22, P. 337], “The adiabatic law  $p = p_0(n/n_0)$  can be assumed, but the customary acoustic value  $\gamma = \frac{5}{3}$  for a gas of particles with 3 external, but no internal, degrees of freedom is not valid. The reason is that the frequency of the present density oscillations is much higher than the collision frequency, contrary to the acoustical limit. Consequently the one-dimensional nature of the density oscillations is maintained. A value of  $\gamma$  appropriate to 1 translational degree of freedom must be used. Since  $\gamma = (m+2)/m$ , where  $m$  is the number of degrees of freedom, we have in this case  $\gamma = 3$ .” We therefore concentrate in this paper on this most significant physical case, and assume the pressure is given by

$$p(n) = \frac{1}{3}n^3. \tag{1.2}$$

In 1998, Guo in [7] first studied Euler-Poisson system in three dimensional case. He observed that the linearized Euler-Poisson system for the electron fluid is the Klein-Gordon equation, due to plasma oscillations created by the electric field, and constructed the smooth irrotational solutions with small amplitude for all time (never develop shocks). This is a very surprising result compared to the work of Sideris [27] for pure Euler equations, where the solutions will blow up even under small perturbations. It is the dispersive effect of the electric field that enhances the linear decay rate and prevents shock formation. Note that the decay rate in the  $L^\infty - L^1$  decay estimate for the linear Klein-Gordon equation is  $t^{-\frac{d}{2}}$ , which is integrable when  $d = 3$ .

In lower dimension case (1D and 2D case), as the decay rate for the linearized Euler-Poisson equations is worse than 3D case, so that the construction of global smooth solution is much more challenging. In 2D case, the decay rate in the  $L^\infty - L^1$  decay estimate is the borderline  $t^{-1}$ , so the main obstructions in the 2D Euler-Poisson system are slow (non-integrable) dispersion. Recently, smooth irrotational solutions for the 2D Euler-Poisson system (1.1) are constructed independently in [16, 25]. See also [23, 24] for related results on two dimensional case.

Such an unexpected and subtle dispersive effect has been discovered and exploited in other two-fluid models, which leads to persistence of global smooth solutions and absence of shock formations. Among the results, we refer to [3, 4, 5, 8, 9, 10, 17, 15].

It has remained as an outstanding question about whether or not shock formations can be suppressed in 1D for any two-fluid model. For the Euler-Poisson system (1.1), the linear time decay rate is merely of  $t^{-1/2}$ , and even for general 1D scalar nonlinear Klein-Gordon equation, singularity (shock waves) might develop for small initial data [14]. Nevertheless, we settle this question in affirmative for the Euler-Poisson system (1.1) with  $\gamma = 3$  by constructing global smooth solutions with small amplitude. To state precisely our result, we set all the physical constants  $m_e$ ,  $e$ ,  $n_0$  and  $4\pi$  to be one. From (1.2), system (1.1) reduces to

$$\begin{aligned} n_t + (nv)_x &= 0, \\ v_t + vv_x + nn_x &= \psi_x, \\ \psi_{xx} &= n - 1. \end{aligned} \tag{1.3}$$

Moreover, if  $E := \psi_x$ , system (1.3) can be further rewritten as

$$\begin{aligned} E_t + v + vE_x &= 0, \\ v_t + E_{xx} - E + vv_x + E_x E_{xx} &= 0. \end{aligned} \tag{1.4}$$

From now on, we mainly focus on the above system. Let

$$r := \frac{E}{2}, \quad u := -\frac{v}{2\langle \partial_x \rangle}, \tag{1.5}$$

where  $\langle \partial_x \rangle := \sqrt{1 - \partial_x^2}$ , then (1.4) can be written in an equivalent form

$$\begin{pmatrix} r \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & -\langle \partial_x \rangle \\ \langle \partial_x \rangle & 0 \end{pmatrix} \begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} 2\langle \partial_x \rangle u & r_x \\ \frac{\partial_x}{\langle \partial_x \rangle} [(\langle \partial_x \rangle u)^2 + (r_x)^2] & \end{pmatrix}. \quad (1.6)$$

Once we obtain global smooth solutions  $(r, u)$  for system (1.6), then we also obtain smooth solutions  $(E, v)$  for system (1.4) by the relation (1.5), and thus the density  $n$  in (1.3) is  $n = 1 + \psi_{xx} = 1 + E_x$ .

The main result of the paper is stated in the following theorem.

**Theorem 1.1** *Let  $N = 300$ ,  $N_1 = 15$ ,  $0 < p_0 < 10^{-3}$ ,  $U := (r, u)^T$  and  $\Gamma := t\partial_x + x\partial_t$ . Then there exists  $\epsilon_0 = \epsilon_0(p_0) > 0$  sufficiently small such that if*

$$\|U(0)\|_{H^N} + \|xU(0)\|_{H^{N_1+1}} + \|\langle \xi \rangle^{N_1+10} \widehat{U}(0)\|_{L^\infty} \leq \epsilon_0, \quad (1.7)$$

the system (1.6) admits a global solution  $U \in C(\mathbb{R}^+; H^N)$  satisfying

$$\sup_{t>0} [(1+t)^{-p_0} \|U(t)\|_{H^N} + (1+t)^{-p_0} \|\Gamma U(t)\|_{H^{N_1}} + (1+t)^{1/2} \|U(t)\|_{W^{N_1+10,\infty}}] \lesssim \epsilon_0.$$

We remark that (1.7) implies the neutrality condition

$$\int_{\mathbb{R}} (n(0, x) - 1) dx = 0,$$

which is conserved for all time. The above theorem shows that under small perturbations around the equilibrium, system (1.3) still has a global smooth solution. However, unlike the 2D or 3D case, we can not obtain the usual scattering result for 1D Euler-Poisson system. Instead, we will see that solutions approach to a nonlinear asymptotic state. To show this phenomenon, we set

$$h = \frac{1}{2}E - \frac{i}{2\langle \partial_x \rangle}v = r + iu, \quad (1.8)$$

then system (1.6) is equivalent to the following complex-valued Klein-Gordon equation

$$h_t + i\langle \partial_x \rangle h = \frac{1}{2i}(h + \bar{h})_x \langle \partial_x \rangle (h - \bar{h}) + \frac{\partial_x}{4i\langle \partial_x \rangle} [\langle \partial_x \rangle (h - \bar{h})]^2 - \frac{\partial_x}{4i\langle \partial_x \rangle} [(h + \bar{h})_x]^2. \quad (1.9)$$

By Shatah's normal form transformation [26], we may make a change of new unknown  $g$  (see (4.3)) such that

$$g_t + i\langle \partial_x \rangle g = \mathcal{N}(h), \quad (1.10)$$

where the cubic term  $\mathcal{N}(h)$  is given in (4.6). In this work, we show that there exists a unique  $w_\infty(\xi) \in L^\infty$  such that

$$\sup_{t \geq 0} [(1+t)^\delta \|\langle \xi \rangle^{N_1+10} e^{i\vartheta(t,\xi)} \widehat{w}(t, \xi) - w_\infty(\xi)\|_{L^\infty}] \lesssim \epsilon_0$$

for some  $\delta > 0$ , where  $w := e^{it\langle \partial_x \rangle} g$  is the linear profile of  $g$ , and  $\vartheta$  is a real-valued function defined by (4.37). This result says the solution of the equation (1.9) tends to a nonlinear asymptotic state as time goes to infinity, thus such equation possesses a modified scattering behavior. Therefore, we extend the previous work on electron type Euler-Poisson system to one dimensional case. Together with the work [7, 16, 25], our result provides a complete picture of Klein-Gordon effect which prevents formation of small shocks in all physical dimensions for the Euler-Poisson system (1.1). Moreover, even though (1.1) is a hyperbolic system of conservation (balance) law [2], the construction of its global BV solutions with small amplitude (hence uniqueness) has remained outstanding. Our result also demonstrates that the standard 1D BV theory is not needed for small smooth initial data for (1.1), and it is ill-suited to capture the delicate dispersive Klein-Gordon effect which prevents the shock formation.

Our work is inspired by recent work of [1, 19, 20, 21] on water waves system, which depends on a delicate interplay between higher energy estimates and a low order  $L^\infty$  estimate. It is well-known that due to poor decay rate of  $t^{-1/2}$ , the classical energy estimate with quadratic nonlinearity is impossible to close, and it is necessary to perform the energy estimate in a new system with a *cubic* nonlinearity. In other words, one would wish to make an “energy normal form” transformation in the energy estimate. Unfortunately, Shatah’s normal form transformation introduces “loss” of derivatives. Even though it is sufficient for lower order  $L^\infty$  decay estimates, it is in general not compatible for high order energy estimate. As a matter of fact, such an “energy normal form” may not exist for general 1D quasi-linear Klein-Gordon equations.

Our first important step is the construction of an “energy normal form” transformation in Section 2. We follow the procedure in [1], and the special structure with  $\gamma = 3$  enables us to discover subtle cancelations for the part of the quadratic terms  $Q - B$  (see Proposition 2.2), during the Sobolev energy estimates. Meanwhile, we construct normal form transformations without “loss” of derivatives, which eliminates the other part of quadratic terms  $B$  (see Proposition 2.5). In Section 2.4, we complete the whole process of higher order energy estimates (Proposition 2.1).

For the  $L^\infty$  decay estimate, we employ the following refined linear decay estimate for the solution  $g$  (see Lemma A.1),

$$\|g\|_{L^\infty} \lesssim (1+t)^{-\frac{1}{2}} \|\widehat{w}\|_{L^\infty} + (1+t)^{-\frac{5}{8}} (\|w\|_{H^2} + \|xw\|_{H^1}), \quad \forall t \geq 0, \quad (1.11)$$

where  $w = e^{it\langle \partial_x \rangle} g$ . It is important to note that  $-\frac{5}{8} < -\frac{1}{2}$ , so there is room for mild growth for  $\|w\|_{H^2}$  and  $\|xw\|_{H^1}$ . Then it reduces to low order estimates for  $\|xw\|_{H^{N_1}}$  and  $\|\langle \xi \rangle^{N_1+10} \widehat{w}\|_{L^\infty}$ , respectively.

The second important step is to estimate  $\|xw\|_{H^{N_1}}$ . In the work [1] and [19], the crucial homogeneous scaling operator  $S = \frac{1}{2}t\partial_t + x\partial_x$  for the gravity water waves system

is employed. Unfortunately, in our problem, the natural operator<sup>1</sup>  $\tilde{\Gamma} = t\partial_x - i\langle \partial_x \rangle x$  for the Klein-Gordon case, is not homogeneous. So the energy estimate fails for  $\tilde{\Gamma}U$ , as  $\tilde{\Gamma}$  could not commute with the nonlinear terms. Instead, we use the homogeneous vector field operator  $\Gamma = t\partial_x + x\partial_t$  to perform energy estimate for  $\Gamma U$ . The key observation is the following relation between  $\Gamma$  and  $\tilde{\Gamma}$ ,

$$\tilde{\Gamma}g = \Gamma g - x(\partial_t + i\langle \partial_x \rangle)g + \frac{i\partial_x}{\langle \partial_x \rangle}g = \Gamma g - x\mathcal{N}(h) + \frac{i\partial_x}{\langle \partial_x \rangle}g.$$

In Sections 3.1–3.3, we obtain energy estimate for  $\Gamma U$  by applying similar strategy as used in Section 2, see Proposition 3.1. In addition, using this modified normal form process, we also control the low energy estimate for  $xU$  in Section 3.4, which is necessary when estimating the difference between  $\Gamma g$  and  $\tilde{\Gamma}g$ . We establish that  $\|xU\|_{H^{N_1}}$  grows almost linearly

$$\|xU(t)\|_{H^{N_1}} \lesssim (1+t)^{1+p_0}$$

for  $p_0 \ll 1$ , see Proposition 3.2. Thanks again to the *cubic* structure of  $\mathcal{N}(h)$ , it yields that  $\|x\mathcal{N}(h)\|_{H^{N_1}}$  can be bounded by  $(1+t)^{p_0}$ , which is sufficient for our argument. In virtue of the identity

$$\langle \partial_x \rangle (xw) = ie^{it\langle \partial_x \rangle} \tilde{\Gamma}g,$$

we finally can able to control  $\|xw\|_{H^{N_1}}$  via the estimates of  $g$ ,  $\Gamma g$  and  $x\mathcal{N}(h)$ . The details are presented in Section 4.1.

The estimate for  $\|\langle \xi \rangle^{N_1+10}\widehat{w}\|_{L^\infty}$  is carried out in Section 4.2 as an adaptation of the proof in [18, 19, 20, 21]. Through precise frequency decompositions and stationary phase analysis, we notice that a phase correction is needed to the leading order term and thus leads to the modified scattering behavior (Proposition 4.5). Using the above norms and (1.11), we close our decay argument in Section 4.3.

Finally, the global existence result follows from (1.11), Proposition 2.1, Propositions 3.1–3.2 and Proposition 4.1.

### Notations:

- The Fourier transform and Fourier inverse transform are defined by

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx = \widehat{f}(\xi),$$

$$(\mathcal{F}^{-1}g)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} g(\xi) d\xi.$$

- Assume  $f$  is a scalar function,  $V$  is a vector-valued function (or scalar function) and  $M(\xi_1, \xi_2)$  is a matrix symbol (or scalar symbol). Define the bilinear operator

$$\mathcal{O}[f, M]V := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{f}(\xi_1) M(\xi_1, \xi_2) \widehat{V}(\xi_2) d\xi_1 d\xi_2. \quad (1.12)$$

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<sup>1</sup>The operator  $\tilde{\Gamma}$  was used in [12, 13] to study the scattering behavior for cubic and quadratic nonlinear Klein-Gordon equation without derivatives.

- Let  $\varphi \in C_c^\infty(\mathbb{R})$  be a radial function with the properties such that  $0 \leq \varphi \leq 1$ ,  $\varphi(\xi) = 1$  for  $|\xi| \leq 5/4$  and  $\text{supp } \varphi \subset [-8/5, 8/5]$ . Then for  $k \in \mathbb{Z}$ , we write  $\varphi_k(\xi) := \varphi(\xi/2^k) - \varphi(\xi/2^{k-1})$ . The dyadic frequency localization operator  $P_k$  is defined by

$$\widehat{P_k f}(\xi) := \varphi_k(|\xi|) \widehat{f}(\xi).$$

Moreover, for  $a > 0$ , we denote by  $P_{\leq a}$  the projector with symbol  $\varphi(|\xi|/a)$ .

- For any  $\rho \in \{0\} \cup \mathbb{N}$ , we denote by  $C^0(\mathbb{R})$  the space of bounded continuous functions, by  $C^\rho(\mathbb{R})$  the space of  $C^0(\mathbb{R})$  functions whose derivatives of order less or equal to  $\rho$  are in  $C^0(\mathbb{R})$ .
- $\langle \partial_x \rangle := \sqrt{1 - \partial_x^2}$ ,  $\langle \xi \rangle := \sqrt{1 + \xi^2}$ .

## 2 Energy estimate

Our aim in this section is to prove the following energy estimate.

**Proposition 2.1** *Let  $U(t) \in C([0, T]; H^N)$  be the solution of system (1.6). Assume (1.7) holds and*

$$\sup_{t \in [0, T]} [(1+t)^{-p_0} \|U(t)\|_{H^N} + (1+t)^{1/2} \|U(t)\|_{W^{N_1+10, \infty}}] \lesssim \epsilon_1, \quad (2.1)$$

where  $0 < \epsilon_0 \ll \epsilon_1 \ll 1$ ,  $N = 300$ ,  $N_1 = 15$  and  $0 < p_0 < 10^{-3}$ . Then we have

$$\sup_{t \in [0, T]} [(1+t)^{-p_0} \|U(t)\|_{H^N}] \lesssim \epsilon_0 + \epsilon_1^2, \quad (2.2)$$

where the implicit constant depends only on  $p_0$ .

### 2.1 Decomposition of the nonlinear terms

Fix a cut off function  $\theta \in C^\infty(\mathbb{R} \times \mathbb{R})$  satisfying

(1) There exist  $\tilde{\epsilon}_1, \tilde{\epsilon}_2$  such that  $0 < 2\tilde{\epsilon}_1 < \tilde{\epsilon}_2 < 1/2$  and

$$\begin{aligned} \theta(\xi_1, \xi_2) &= 1, & |\xi_1| \leq \tilde{\epsilon}_1 |\xi_2|, \\ \theta(\xi_1, \xi_2) &= 0, & |\xi_1| \geq \tilde{\epsilon}_2 |\xi_2|. \end{aligned} \quad (2.3)$$

(2) For any  $\alpha, \beta \in \mathbb{N} \cup \{0\}$ , there holds

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \theta(\xi_1, \xi_2)| \leq C_{\alpha, \beta} \langle \xi_2 \rangle^{-\alpha-\beta}, \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \quad (2.4)$$

(3)  $\theta$  satisfies the symmetry condition

$$\theta(\xi_1, \xi_2) = \theta(-\xi_1, -\xi_2) = \theta(-\xi_1, \xi_2). \quad (2.5)$$

Then we define the paraproduct  $T_f g$  and the remainder  $R_B(f, g)$  as

$$T_f g := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \theta(\xi_1, \xi_2) \widehat{f}(\xi_1) \widehat{g}(\xi_2) d\xi_1 d\xi_2,$$

$$R_B(f, g) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} (1 - \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1)) \widehat{f}(\xi_1) \widehat{g}(\xi_2) d\xi_1 d\xi_2.$$

With this definition, for any  $f$  and  $g$ , we have the following Bony decomposition

$$fg = T_f g + T_g f + R_B(f, g). \quad (2.6)$$

Let

$$U = \begin{pmatrix} r \\ u \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -\langle \partial_x \rangle \\ \langle \partial_x \rangle & 0 \end{pmatrix}, \quad (2.7)$$

and

$$Q_1(\xi_1, \xi_2) := \begin{pmatrix} 0 & q_1(\xi_1, \xi_2) \\ q_4(\xi_1, \xi_2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i\xi_1 \langle \xi_2 \rangle \\ \frac{-2i(\xi_1 + \xi_2)\xi_1 \xi_2}{\langle \xi_1 + \xi_2 \rangle} & 0 \end{pmatrix} \theta(\xi_1, \xi_2), \quad (2.8)$$

$$Q_2(\xi_1, \xi_2) := \begin{pmatrix} q_2(\xi_1, \xi_2) & 0 \\ 0 & q_3(\xi_1, \xi_2) \end{pmatrix} = \begin{pmatrix} 2i\xi_2 \langle \xi_1 \rangle & 0 \\ 0 & \frac{2i(\xi_1 + \xi_2) \langle \xi_1 \rangle \langle \xi_2 \rangle}{\langle \xi_1 + \xi_2 \rangle} \end{pmatrix} \theta(\xi_1, \xi_2), \quad (2.9)$$

$$S_1(\xi_1, \xi_2) := \begin{pmatrix} 0 & s_1(\xi_1, \xi_2) \\ s_4(\xi_1, \xi_2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\xi_1 \langle \xi_2 \rangle \\ \frac{-i(\xi_1 + \xi_2)\xi_1 \xi_2}{\langle \xi_1 + \xi_2 \rangle} & 0 \end{pmatrix} (1 - \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1)), \quad (2.10)$$

$$S_2(\xi_1, \xi_2) := \begin{pmatrix} s_2(\xi_1, \xi_2) & 0 \\ 0 & s_3(\xi_1, \xi_2) \end{pmatrix} = \begin{pmatrix} i\xi_2 \langle \xi_1 \rangle & 0 \\ 0 & \frac{i(\xi_1 + \xi_2) \langle \xi_1 \rangle \langle \xi_2 \rangle}{\langle \xi_1 + \xi_2 \rangle} \end{pmatrix} (1 - \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1)). \quad (2.11)$$

By (1.12) and (2.6), system (1.6) is then transformed into

$$U_t + DU = \mathcal{O}[r, Q_1]U + \mathcal{O}[u, Q_2]U + \mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U. \quad (2.12)$$

Here,  $Q_1, Q_2$  are the symbols of low-high interaction terms, with one local/global derivative on the function of high frequency, and  $S_1, S_2$  are the symbols of nonlinear terms with high-high interactions. The low-high terms will cause loss of derivatives when performing energy estimate, so we shall do some modifications with these terms, see Proposition 2.2 in the next subsection.

## 2.2 Modifying low-high interaction terms

In this subsection, we prove

**Proposition 2.2** Let  $Q_1, Q_2$  be given by (2.8)–(2.9). Then there exist two matrices  $B_1$  and  $B_2$  with the form

$$B_1 = \begin{pmatrix} 0 & b_1(\xi_1, \xi_2) \\ b_4(\xi_1, \xi_2) & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_2(\xi_1, \xi_2) & 0 \\ 0 & b_3(\xi_1, \xi_2) \end{pmatrix} \quad (2.13)$$

such that

$$\operatorname{Re} \langle \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1]U, \langle \partial_x \rangle^N U \rangle = 0, \quad (2.14)$$

$$\operatorname{Re} \langle \langle \partial_x \rangle^N \mathcal{O}[u, Q_2 - B_2]U, \langle \partial_x \rangle^N U \rangle = 0, \quad (2.15)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $L^2$  space. Moreover, for any  $\alpha, \beta = 0, 1$ ,

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta b_j(\xi_1, \xi_2)| \lesssim \langle \xi_1 \rangle^2, \quad j = 1, 2, 3, 4, \quad (2.16)$$

and for any  $\rho \geq 3$ ,

$$\| \langle \partial_x \rangle^N \mathcal{O}[f, B_1]U \|_{L^2} + \| \langle \partial_x \rangle^N \mathcal{O}[f, B_2]U \|_{L^2} \lesssim \| f \|_{C^\rho} \| U \|_{H^N}. \quad (2.17)$$

To prove this proposition, one should use the following lemma.

**Lemma 2.3** [1] Assume  $f$  is a real-valued function, and  $M$  is a matrix. Then we have

$$(\mathcal{O}[f, M])^* = \mathcal{O}[f, \widetilde{M}], \quad \widetilde{M}(\xi_1, \xi_2) := \overline{M^T(-\xi_1, \xi_1 + \xi_2)},$$

where  $M^T$  is the transpose of  $M$ .

*Proof.* By the definition of  $\mathcal{O}[f, M]W$  (see (1.12)),

$$\mathcal{F}(\mathcal{O}[f, M]W)(\eta) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(\xi) M(\xi, \eta - \xi) \widehat{W}(\eta - \xi) d\xi.$$

Using the fact  $f$  is real-valued, we have

$$\begin{aligned} \langle \mathcal{O}[f, M]^*V, W \rangle &= \langle V, \mathcal{O}[f, M]W \rangle \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \widehat{V}(\eta) \left( \int_{\mathbb{R}} \overline{\widehat{f}(\xi)} M(\xi, \eta - \xi) \widehat{W}(\eta - \xi) d\xi \right) d\eta \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{V}(\xi + \eta) \widehat{f}(-\xi) \overline{M(\xi, \eta)} \overline{\widehat{W}(\eta)} d\xi d\eta \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{2\pi} \left( \int_{\mathbb{R}} \widehat{f}(\xi) \overline{M^T(-\xi, \eta)} \widehat{V}(\eta - \xi) d\xi \right) \overline{\widehat{W}(\eta)} d\eta \\ &= \langle \mathcal{O}[f, \widetilde{M}]V, W \rangle, \end{aligned}$$

from which we can obtain the desired result  $\widetilde{M}(\xi, \eta) = \overline{M^T(-\xi, \xi + \eta)}$ .  $\square$

We will also need the following anisotropic multiplier estimate.

**Lemma 2.4** *There holds*

$$\|\mathcal{O}[f, M]V\|_{L^2(\mathbb{R})} \lesssim \|M(\xi, \eta - \xi)\|_{L_\eta^\infty H_\xi^1} \|f\|_{L^\infty(\mathbb{R})} \|V\|_{L^2(\mathbb{R})}. \quad (2.18)$$

Similar estimates are also used in [10, 11]. The proof of this lemma is given in Lemma B.1 of the appendix.

*Proof of Proposition 2.2.* We rewrite equation (2.14) as

$$\begin{aligned} 0 &= 2\operatorname{Re}\langle \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1]U, \langle \partial_x \rangle^N U \rangle \\ &= \langle \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1]U, \langle \partial_x \rangle^N U \rangle + \overline{\langle \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1]U, \langle \partial_x \rangle^N U \rangle} \\ &= \langle \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1]U, \langle \partial_x \rangle^N U \rangle + \langle \langle \partial_x \rangle^N U, \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1]U \rangle \\ &= \langle \langle \partial_x \rangle^{2N} \mathcal{O}[r, Q_1 - B_1]U, U \rangle + \langle \langle \partial_x \rangle^{2N} U, \mathcal{O}[r, Q_1 - B_1]U \rangle \\ &= \langle \langle \partial_x \rangle^{2N} \mathcal{O}[r, Q_1 - B_1]U, U \rangle + \langle \mathcal{O}[r, Q_1 - B_1]^*(\langle \partial_x \rangle^{2N} U), U \rangle. \end{aligned}$$

In order to prove (2.14), we only need to verify

$$\langle \partial_x \rangle^{2N} \mathcal{O}[r, Q_1]U + \mathcal{O}[r, Q_1]^*(\langle \partial_x \rangle^{2N} U) = \langle \partial_x \rangle^{2N} \mathcal{O}[r, B_1]U + \mathcal{O}[r, B_1]^*(\langle \partial_x \rangle^{2N} U). \quad (2.19)$$

From Lemma 2.3, we see

$$\begin{aligned} \langle \partial_x \rangle^{2N} \mathcal{O}[r, Q_1]U &= \mathcal{O}[r, \langle \xi_1 + \xi_2 \rangle^{2N} Q_1(\xi_1, \xi_2)]U, \\ \mathcal{O}[r, Q_1]^*(\langle \partial_x \rangle^{2N} U) &= \mathcal{O}[r, \langle \xi_2 \rangle^{2N} \overline{Q_1^T(-\xi_1, \xi_1 + \xi_2)}]U. \end{aligned}$$

Define

$$B_1(\xi_1, \xi_2) := \frac{\langle \xi_1 + \xi_2 \rangle^{2N} Q_1(\xi_1, \xi_2) + \langle \xi_2 \rangle^{2N} \overline{Q_1^T(-\xi_1, \xi_1 + \xi_2)}}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}}, \quad (2.20)$$

then  $B_1(\xi_1, \xi_2) = \overline{B_1^T(-\xi_1, \xi_1 + \xi_2)}$ , and by Lemma 2.3, we have  $\mathcal{O}[r, B_1] = \mathcal{O}[r, B_1]^*$ . With such choice of  $B_1(\xi_1, \xi_2)$ , the identity (2.19) holds, and (2.14) thus follows.

Similarly, in order to prove (2.15), we only need to show

$$\langle \partial_x \rangle^{2N} \mathcal{O}[u, Q_2]U + \mathcal{O}[u, Q_2]^*(\langle \partial_x \rangle^{2N} U) = \langle \partial_x \rangle^{2N} \mathcal{O}[u, B_2]U + \mathcal{O}[u, B_2]^*(\langle \partial_x \rangle^{2N} U).$$

Define

$$B_2(\xi_1, \xi_2) := \frac{\langle \xi_1 + \xi_2 \rangle^{2N} Q_2(\xi_1, \xi_2) + \langle \xi_2 \rangle^{2N} \overline{Q_2^T(-\xi_1, \xi_1 + \xi_2)}}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}}, \quad (2.21)$$

then we can check  $\mathcal{O}[u, B_2] = \mathcal{O}[u, B_2]^*$  and (2.15) holds.

In order to prove (2.16), we should calculate  $b_j(\xi_1, \xi_2)$  ( $j = 1, 2, 3, 4$ ) carefully. From (2.13), (2.20), (2.8) and (2.5), the expression for  $b_1(\xi_1, \xi_2)$  can be written as exactly as

$$b_1(\xi_1, \xi_2)$$

$$\begin{aligned}
&= \frac{2i\xi_1}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}} \left[ \langle \xi_1 + \xi_2 \rangle^{2N} \langle \xi_2 \rangle \theta(\xi_1, \xi_2) - \langle \xi_2 \rangle^{2N} \frac{\xi_2(\xi_1 + \xi_2)}{\langle \xi_2 \rangle} \theta(-\xi_1, \xi_1 + \xi_2) \right] \\
&= \frac{2i\xi_1}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}) \langle \xi_2 \rangle} \left[ \langle \xi_1 + \xi_2 \rangle^{2N} \langle \xi_2 \rangle^2 \theta(\xi_1, \xi_2) - \langle \xi_2 \rangle^{2N} \xi_2(\xi_1 + \xi_2) \theta(\xi_1, \xi_1 + \xi_2) \right] \\
&= \frac{2i\xi_1 \xi_2}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}) \langle \xi_2 \rangle} \left[ (\xi_1 + \xi_2)^{2N} \xi_2 \theta(\xi_1, \xi_2) - (\xi_1 + \xi_2) \xi_2^{2N} \theta(\xi_1, \xi_1 + \xi_2) \right] + r_1(\xi_1, \xi_2) \\
&= \frac{2i\xi_1 \xi_2}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}) \langle \xi_2 \rangle} \chi(\xi_1, \xi_2) + r_1(\xi_1, \xi_2),
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
\chi(\xi_1, \xi_2) &:= (\xi_1 + \xi_2)^{2N} \xi_2 \theta(\xi_1, \xi_2) - (\xi_1 + \xi_2) \xi_2^{2N} \theta(\xi_1, \xi_1 + \xi_2), \\
r_1(\xi_1, \xi_2) &:= \frac{2i\xi_1 \theta(\xi_1, \xi_2)}{\langle \xi_2 \rangle (\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})} [\langle \xi_1 + \xi_2 \rangle^{2N} \langle \xi_2 \rangle^2 - (\xi_1 + \xi_2)^{2N} \xi_2^2] \\
&\quad - \frac{2i\xi_1 \xi_2 (\xi_1 + \xi_2) \theta(\xi_1, \xi_1 + \xi_2)}{\langle \xi_2 \rangle (\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})} [\langle \xi_2 \rangle^{2N} - \xi_2^{2N}].
\end{aligned} \tag{2.23}$$

We decompose  $\chi(\xi_1, \xi_2)$  into  $I_1 + I_2$  with

$$\begin{aligned}
I_1 &:= ((\xi_1 + \xi_2)^{2N} \xi_2 - (\xi_1 + \xi_2) \xi_2^{2N}) \theta(\xi_1, \xi_2), \\
I_2 &:= (\xi_1 + \xi_2) \xi_2^{2N} (\theta(\xi_1, \xi_2) - \theta(\xi_1, \xi_1 + \xi_2)).
\end{aligned}$$

Recall the bound (2.4). For  $I_1$ , it is easy to see

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta I_1| \lesssim |\xi_1| |\xi_2|^{2N}, \text{ where } |\xi_1| \ll |\xi_2|, \quad \alpha, \beta = 0, 1. \tag{2.24}$$

For  $I_2$ , note that  $\theta(\xi_1, \xi_2) - \theta(\xi_1, \xi_1 + \xi_2) \neq 0$  implies  $|\xi_1| \sim |\xi_2|$ , then

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta I_2| \lesssim |\xi_1| |\xi_2|^{2N}, \quad \alpha, \beta = 0, 1. \tag{2.25}$$

Also, a direct computation shows that the remainder  $r_1(\xi_1, \xi_2)$  satisfies

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta r_1(\xi_1, \xi_2)| \lesssim \langle \xi_1 \rangle \langle \xi_2 \rangle^{-1}, \quad \alpha, \beta = 0, 1. \tag{2.26}$$

Therefore, we conclude from (2.22)–(2.26) that (2.16) holds for  $j = 1$ .

Similarly, using (2.20), (2.21), (2.8) and (2.9), the expressions for  $b_4(\xi_1, \xi_2)$ ,  $b_2(\xi_1, \xi_2)$ ,  $b_3(\xi_1, \xi_2)$  in (2.13) are

$$\begin{aligned}
b_4(\xi_1, \xi_2) &:= \frac{2i\xi_1}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}} \left[ -\langle \xi_1 + \xi_2 \rangle^{2N} \frac{\xi_2(\xi_1 + \xi_2)}{\langle \xi_1 + \xi_2 \rangle} \theta(\xi_1, \xi_2) \right. \\
&\quad \left. + \langle \xi_2 \rangle^{2N} \langle \xi_1 + \xi_2 \rangle \theta(\xi_1, \xi_1 + \xi_2) \right], \\
b_2(\xi_1, \xi_2) &:= \frac{2i\langle \xi_1 \rangle}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}} [\langle \xi_1 + \xi_2 \rangle^{2N} \xi_2 \theta(\xi_1, \xi_2) - \langle \xi_2 \rangle^{2N} (\xi_1 + \xi_2) \theta(\xi_1, \xi_1 + \xi_2)], \\
b_3(\xi_1, \xi_2) &:= \frac{2i\langle \xi_1 \rangle}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}} \left[ \langle \xi_1 + \xi_2 \rangle^{2N} \frac{(\xi_1 + \xi_2) \langle \xi_2 \rangle}{\langle \xi_1 + \xi_2 \rangle} \theta(\xi_1, \xi_2) \right]
\end{aligned}$$

$$- \langle \xi_2 \rangle^{2N} \frac{\xi_2(\xi_1 + \xi_2)}{\langle \xi_2 \rangle} \theta(\xi_1, \xi_1 + \xi_2) \Big].$$

Using the function  $\chi(\xi_1, \xi_2)$  (see (2.23)), we have

$$\begin{aligned} b_4(\xi_1, \xi_2) &= \frac{-2i\xi_1(\xi_1 + \xi_2)}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})\langle \xi_1 + \xi_2 \rangle} \chi(\xi_1, \xi_2) + r_4(\xi_1, \xi_2), \\ b_2(\xi_1, \xi_2) &= \frac{2i\langle \xi_1 \rangle}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}} \chi(\xi_1, \xi_2) + r_2(\xi_1, \xi_2), \\ b_3(\xi_1, \xi_2) &= \frac{2i\langle \xi_1 \rangle(\xi_1 + \xi_2)\xi_2}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})\langle \xi_1 + \xi_2 \rangle\langle \xi_2 \rangle} \chi(\xi_1, \xi_2) + r_3(\xi_1, \xi_2), \end{aligned} \quad (2.27)$$

where

$$\begin{aligned} r_4(\xi_1, \xi_2) &= \frac{2i\xi_1\xi_2(\xi_1 + \xi_2)\theta(\xi_1, \xi_2)}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})\langle \xi_1 + \xi_2 \rangle} [(\xi_1 + \xi_2)^{2N} - \langle \xi_1 + \xi_2 \rangle^{2N}] \\ &\quad + \frac{2i\xi_1\theta(\xi_1, \xi_1 + \xi_2)}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})\langle \xi_1 + \xi_2 \rangle} [\langle \xi_2 \rangle^{2N}\langle \xi_1 + \xi_2 \rangle^2 - \xi_2^{2N}(\xi_1 + \xi_2)^2], \\ r_2(\xi_1, \xi_2) &= \frac{2i\langle \xi_1 \rangle\xi_2\theta(\xi_1, \xi_2)}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}} [(\xi_1 + \xi_2)^{2N} - (\xi_1 + \xi_2)^{2N}] \\ &\quad - \frac{2i\langle \xi_1 \rangle(\xi_1 + \xi_2)\theta(\xi_1, \xi_1 + \xi_2)}{\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N}} [\langle \xi_2 \rangle^{2N} - \xi_2^{2N}], \\ r_3(\xi_1, \xi_2) &= \frac{2i\langle \xi_1 \rangle(\xi_1 + \xi_2)\theta(\xi_1, \xi_2)}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})\langle \xi_1 + \xi_2 \rangle\langle \xi_2 \rangle} [(\xi_1 + \xi_2)^{2N}\langle \xi_2 \rangle^2 - (\xi_1 + \xi_2)^{2N}\xi_2^2] \\ &\quad - \frac{2i\langle \xi_1 \rangle\xi_2\theta(\xi_1, \xi_1 + \xi_2)}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})\langle \xi_1 + \xi_2 \rangle\langle \xi_2 \rangle} [\langle \xi_2 \rangle^{2N}\langle \xi_1 + \xi_2 \rangle^2 - \xi_2^{2N}(\xi_1 + \xi_2)^2]. \end{aligned}$$

With similar argument as above, we can obtain, for any  $\alpha, \beta = 0, 1$  and  $j = 2, 3, 4$ ,

$$\begin{aligned} |\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta r_j(\xi_1, \xi_2)| &\lesssim \langle \xi_1 \rangle \langle \xi_2 \rangle^{-1}, \\ |\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta b_j(\xi_1, \xi_2)| &\lesssim \langle \xi_1 \rangle^2. \end{aligned} \quad (2.28)$$

Therefore, the estimate (2.16) holds for all  $j$ ,  $j = 1, 2, 3, 4$ .

Note that

$$\begin{aligned} \langle \partial_x \rangle^N \mathcal{O}[f, B_1]U &= \frac{1}{(2\pi)^2} \langle \partial_x \rangle^N \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{f}(\xi_1) B_1(\xi_1, \xi_2) \widehat{U}(\xi_2) d\xi_1 d\xi_2 \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{\langle \partial_x \rangle^\rho f}(\xi_1) M(\xi_1, \xi_2) \widehat{\langle \partial_x \rangle^N U}(\xi_2) d\xi_1 d\xi_2 \end{aligned}$$

with

$$M(\xi_1, \xi_2) = \frac{\langle \xi_1 + \xi_2 \rangle^N}{\langle \xi_1 \rangle^\rho \langle \xi_2 \rangle^N} B_1(\xi_1, \xi_2), \quad |\xi_1| \ll |\xi_2|.$$

In view of (2.16), we have for  $\rho \geq 3$ ,

$$\|M(\xi_1, \xi_2 - \xi_1)\|_{L_{\xi_2}^\infty H_{\xi_1}^1} = \left\| \frac{\langle \xi_2 \rangle^N}{\langle \xi_1 \rangle^\rho \langle \xi_2 - \xi_1 \rangle^N} B_1(\xi_1, \xi_2 - \xi_1) \right\|_{L_{\xi_2}^\infty H_{\xi_1}^1} \lesssim \|\langle \xi_1 \rangle^{2-\rho}\|_{L^2} \lesssim 1.$$

So applying Lemma 2.4 yields

$$\|\langle \partial_x \rangle^N \mathcal{O}[f, B_1]U\|_{L^2} \lesssim \|f\|_{C^\rho} \|U\|_{H^N}.$$

Similarly, we can prove

$$\|\langle \partial_x \rangle^N \mathcal{O}[f, B_2]U\|_{L^2} \lesssim \|f\|_{C^\rho} \|U\|_{H^N}.$$

Hence, (2.17) follows from these two estimates. This ends the proof of Proposition 2.2.  $\square$

### 2.3 Energy normal form transformation

For the equation (2.12),

$$U_t + DU = \mathcal{O}[r, Q_1]U + \mathcal{O}[u, Q_2]U + \mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U, \quad (2.29)$$

from Proposition 2.2, we notice that the low-high term  $\mathcal{O}[r, B_1]U + \mathcal{O}[u, B_2]U$ , which is a part of  $\mathcal{O}[r, Q_1]U + \mathcal{O}[u, Q_2]U$ , will not lead to loss of derivatives. Now we can use Shatah's normal form method to eliminate this quadratic term.

**Proposition 2.5** *There exist two matrices  $A_1$  and  $A_2$  defined by*

$$A_1 = \begin{pmatrix} 0 & a_1(\xi_1, \xi_2) \\ a_4(\xi_1, \xi_2) & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2(\xi_1, \xi_2) & 0 \\ 0 & a_3(\xi_1, \xi_2) \end{pmatrix} \quad (2.30)$$

such that

$$\begin{aligned} D\mathcal{O}[r, A_2]U - \mathcal{O}[\langle \partial_x \rangle r, A_1]U - \mathcal{O}[r, A_2]DU &= -\mathcal{O}[r, B_1]U, \\ D\mathcal{O}[u, A_1]U + \mathcal{O}[\langle \partial_x \rangle u, A_2]U - \mathcal{O}[u, A_1]DU &= -\mathcal{O}[u, B_2]U. \end{aligned} \quad (2.31)$$

Moreover, for any  $\alpha, \beta = 0, 1$ , we have

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta a_j(\xi_1, \xi_2)| \lesssim \langle \xi_1 \rangle^3, \quad j = 1, 2, 3, 4. \quad (2.32)$$

*Proof.* Inserting (2.30) into (2.31), we have

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{r}(\xi_1) \left[ \begin{pmatrix} 0 & -\langle \xi_1 + \xi_2 \rangle a_3(\xi_1, \xi_2) \\ \langle \xi_1 + \xi_2 \rangle a_2(\xi_1, \xi_2) & 0 \end{pmatrix} \right. \\ &\quad - \begin{pmatrix} 0 & \langle \xi_1 \rangle a_1(\xi_1, \xi_2) \\ \langle \xi_1 \rangle a_4(\xi_1, \xi_2) & 0 \end{pmatrix} \\ &\quad \left. - \begin{pmatrix} 0 & -\langle \xi_2 \rangle a_2(\xi_1, \xi_2) \\ \langle \xi_2 \rangle a_3(\xi_1, \xi_2) & 0 \end{pmatrix} \right] \widehat{U}(\xi_2) d\xi_1 d\xi_2 \\ &= \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{r}(\xi_1) \begin{pmatrix} 0 & -b_1(\xi_1, \xi_2) \\ -b_4(\xi_1, \xi_2) & 0 \end{pmatrix} \widehat{U}(\xi_2) d\xi_1 d\xi_2. \end{aligned}$$

Similarly,

$$\begin{aligned}
& \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{u}(\xi_1) \left[ \begin{pmatrix} -\langle \xi_1 + \xi_2 \rangle a_4(\xi_1, \xi_2) & 0 \\ 0 & \langle \xi_1 + \xi_2 \rangle a_1(\xi_1, \xi_2) \end{pmatrix} \right. \\
& \quad + \begin{pmatrix} \langle \xi_1 \rangle a_2(\xi_1, \xi_2) & 0 \\ 0 & \langle \xi_1 \rangle a_3(\xi_1, \xi_2) \end{pmatrix} \\
& \quad \left. - \begin{pmatrix} \langle \xi_2 \rangle a_1(\xi_1, \xi_2) & 0 \\ 0 & -\langle \xi_2 \rangle a_4(\xi_1, \xi_2) \end{pmatrix} \right] \widehat{U}(\xi_2) d\xi_1 d\xi_2 \\
& = \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{u}(\xi_1) \begin{pmatrix} -b_2(\xi_1, \xi_2) & 0 \\ 0 & -b_3(\xi_1, \xi_2) \end{pmatrix} \widehat{U}(\xi_2) d\xi_1 d\xi_2.
\end{aligned}$$

Thus we obtain linear equations

$$\begin{pmatrix} -\langle \xi_1 \rangle & \langle \xi_2 \rangle & -\langle \xi_1 + \xi_2 \rangle & 0 \\ 0 & \langle \xi_1 + \xi_2 \rangle & -\langle \xi_2 \rangle & -\langle \xi_1 \rangle \\ -\langle \xi_2 \rangle & \langle \xi_1 \rangle & 0 & -\langle \xi_1 + \xi_2 \rangle \\ \langle \xi_1 + \xi_2 \rangle & 0 & \langle \xi_1 \rangle & \langle \xi_2 \rangle \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -b_1(\xi_1, \xi_2) \\ -b_4(\xi_1, \xi_2) \\ -b_2(\xi_1, \xi_2) \\ -b_3(\xi_1, \xi_2) \end{pmatrix}. \quad (2.33)$$

The solution of the above system is

$$\begin{aligned}
a_1 &= \frac{1}{G} [(-\langle \xi_1 \rangle^2 + \langle \xi_2 \rangle^2 + \langle \xi_1 + \xi_2 \rangle^2) \cdot (\langle \xi_1 \rangle b_1 - \langle \xi_2 \rangle b_2 + \langle \xi_1 + \xi_2 \rangle b_3) \\
&\quad - 2\langle \xi_2 \rangle \langle \xi_1 + \xi_2 \rangle \cdot (\langle \xi_1 \rangle b_4 + \langle \xi_2 \rangle b_3 - \langle \xi_1 + \xi_2 \rangle b_2)], \\
a_2 &= \frac{1}{G} [(\langle \xi_1 \rangle^2 - \langle \xi_2 \rangle^2 - \langle \xi_1 + \xi_2 \rangle^2) \cdot (\langle \xi_1 \rangle b_2 - \langle \xi_2 \rangle b_1 - \langle \xi_1 + \xi_2 \rangle b_4) \\
&\quad - 2\langle \xi_2 \rangle \langle \xi_1 + \xi_2 \rangle \cdot (\langle \xi_1 \rangle b_3 + \langle \xi_2 \rangle b_4 + \langle \xi_1 + \xi_2 \rangle b_1)], \\
a_3 &= \frac{1}{G} [(\langle \xi_1 \rangle^2 - \langle \xi_2 \rangle^2 - \langle \xi_1 + \xi_2 \rangle^2) \cdot (\langle \xi_1 \rangle b_3 + \langle \xi_2 \rangle b_4 + \langle \xi_1 + \xi_2 \rangle b_1) \\
&\quad - 2\langle \xi_2 \rangle \langle \xi_1 + \xi_2 \rangle \cdot (\langle \xi_1 \rangle b_2 - \langle \xi_2 \rangle b_1 - \langle \xi_1 + \xi_2 \rangle b_4)], \\
a_4 &= \frac{1}{G} [(-\langle \xi_1 \rangle^2 + \langle \xi_2 \rangle^2 + \langle \xi_1 + \xi_2 \rangle^2) \cdot (\langle \xi_1 \rangle b_4 + \langle \xi_2 \rangle b_3 - \langle \xi_1 + \xi_2 \rangle b_2) \\
&\quad - 2\langle \xi_2 \rangle \langle \xi_1 + \xi_2 \rangle \cdot (\langle \xi_1 \rangle b_1 - \langle \xi_2 \rangle b_2 + \langle \xi_1 + \xi_2 \rangle b_3)],
\end{aligned} \quad (2.34)$$

where  $G = 2\xi_1^2 + 2\xi_2^2 + 2(\xi_1 + \xi_2)^2 + 3 > 0$ .

Now we prove (2.32). To this end, we first claim that, for any  $\alpha, \beta = 0, 1$ ,

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (\langle \xi_1 \rangle b_j(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle^3, \quad j = 1, 2, 3, 4, \quad (2.35)$$

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (\langle \xi_1 + \xi_2 \rangle b_3(\xi_1, \xi_2) - \langle \xi_2 \rangle b_2(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle^3, \quad (2.36)$$

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (\langle \xi_2 \rangle b_3(\xi_1, \xi_2) - \langle \xi_1 + \xi_2 \rangle b_2(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle^3, \quad (2.37)$$

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (\langle \xi_1 + \xi_2 \rangle b_1(\xi_1, \xi_2) + \langle \xi_2 \rangle b_4(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle^3, \quad (2.38)$$

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (\langle \xi_2 \rangle b_1(\xi_1, \xi_2) + \langle \xi_1 + \xi_2 \rangle b_4(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle^3. \quad (2.39)$$

Indeed, the bound (2.35) is a direct consequence of (2.16). The proofs for (2.36)–(2.39) are similar, so we only show (2.36). From (2.27), we see

$$\begin{aligned} \langle \xi_1 + \xi_2 \rangle b_3(\xi_1, \xi_2) - \langle \xi_2 \rangle b_2(\xi_1, \xi_2) &= \frac{2i\langle \xi_1 \rangle ((\xi_1 + \xi_2)\xi_2 - \langle \xi_2 \rangle^2)}{(\langle \xi_1 + \xi_2 \rangle^{2N} + \langle \xi_2 \rangle^{2N})\langle \xi_2 \rangle} \chi(\xi_1, \xi_2) \\ &\quad + \langle \xi_1 + \xi_2 \rangle r_3(\xi_1, \xi_2) - \langle \xi_2 \rangle r_2(\xi_1, \xi_2). \end{aligned} \quad (2.40)$$

Remember that  $|\xi_1| \ll |\xi_2|$ . The bounds (2.24) and (2.25) imply

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \chi(\xi_1, \xi_2)| \lesssim \langle \xi_1 \rangle \langle \xi_2 \rangle^{2N}, \quad \alpha, \beta = 0, 1. \quad (2.41)$$

Also, using (2.28), we have

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (\langle \xi_1 + \xi_2 \rangle r_3(\xi_1, \xi_2))| + |\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (\langle \xi_2 \rangle r_2(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle, \quad \alpha, \beta = 0, 1. \quad (2.42)$$

Inserting the bounds (2.41)–(2.42) into (2.40), we can obtain (2.36) as desired. Then from (2.34) and the bounds (2.35)–(2.39), it is easy to see (2.32) holds. This completes the proof of Proposition 2.5.  $\square$

Similarly, we also use normal form method to cancel the high-high quadratic term  $\mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U$ . More precisely, we have

**Proposition 2.6** *There exist two matrices  $C_1$  and  $C_2$  defined by*

$$C_1 = \begin{pmatrix} 0 & c_1(\xi_1, \xi_2) \\ c_4(\xi_1, \xi_2) & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} c_2(\xi_1, \xi_2) & 0 \\ 0 & c_3(\xi_1, \xi_2) \end{pmatrix} \quad (2.43)$$

such that

$$\begin{aligned} D\mathcal{O}[r, C_2]U - \mathcal{O}[\langle \partial_x \rangle r, C_1]U - \mathcal{O}[r, C_2]DU &= -\mathcal{O}[r, S_1]U, \\ D\mathcal{O}[u, C_1]U + \mathcal{O}[\langle \partial_x \rangle u, C_2]U - \mathcal{O}[u, C_1]DU &= -\mathcal{O}[u, S_2]U. \end{aligned} \quad (2.44)$$

Moreover, for any  $\alpha, \beta = 0, 1$ , we have

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta c_j| \lesssim \langle \xi_1 \rangle^3, \quad j = 1, 2, 3, 4. \quad (2.45)$$

*Proof.* From (2.43)–(2.44), we obtain linear equations for  $c_1, c_2, c_3$  and  $c_4$

$$\begin{pmatrix} -\langle \xi_1 \rangle & \langle \xi_2 \rangle & -\langle \xi_1 + \xi_2 \rangle & 0 \\ 0 & \langle \xi_1 + \xi_2 \rangle & -\langle \xi_2 \rangle & -\langle \xi_1 \rangle \\ -\langle \xi_2 \rangle & \langle \xi_1 \rangle & 0 & -\langle \xi_1 + \xi_2 \rangle \\ \langle \xi_1 + \xi_2 \rangle & 0 & \langle \xi_1 \rangle & \langle \xi_2 \rangle \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_4 \\ s_2 \\ s_3 \end{pmatrix},$$

where the definitions of  $s_1, s_2, s_3$  and  $s_4$  are given by (2.10)–(2.11). Clearly, the solution  $(c_1, c_2, c_3, c_4)$  is given by replacing  $b_j$  with  $s_j$  ( $j = 1, 2, 3, 4$ ) in (2.34). Moreover, in the support of  $1 - \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1)$ , there holds  $|\xi_1| \sim |\xi_2|$ . Hence, with similar argument as the proof of Proposition 2.5, the bound (2.45) can be obtained easily.  $\square$

Now we define the energy normal form transformation

$$\Phi := U + \mathcal{O}[u, A_1]U + \mathcal{O}[r, A_2]U + \mathcal{O}[u, C_1]U + \mathcal{O}[r, C_2]U, \quad (2.46)$$

so that

$$\begin{aligned} \Phi_t + D\Phi &= U_t + DU + D\mathcal{O}[u, A_1]U + D\mathcal{O}[r, A_2]U + D\mathcal{O}[u, C_1]U + D\mathcal{O}[r, C_2]U \\ &\quad + \mathcal{O}[u_t, A_1]U + \mathcal{O}[u, A_1]U_t + \mathcal{O}[r_t, A_2]U + \mathcal{O}[r, A_2]U_t \\ &\quad + \mathcal{O}[u_t, C_1]U + \mathcal{O}[u, C_1]U_t + \mathcal{O}[r_t, C_2]U + \mathcal{O}[r, C_2]U_t. \end{aligned}$$

Moreover, by (2.12), (1.6), (2.31) and (2.44), we have

$$\begin{aligned} \Phi_t + D\Phi &= \mathcal{O}[r, Q_1]U + D\mathcal{O}[r, A_2]U - \mathcal{O}[r, A_2]DU - \mathcal{O}[\langle \partial_x \rangle r, A_1]U \\ &\quad + \mathcal{O}[u, Q_2]U + D\mathcal{O}[u, A_1]U - \mathcal{O}[u, A_1]DU + \mathcal{O}[\langle \partial_x \rangle u, A_2]U \\ &\quad + \mathcal{O}[r, S_1]U + D\mathcal{O}[r, C_2]U - \mathcal{O}[r, C_2]DU - \mathcal{O}[\langle \partial_x \rangle r, C_1]U \\ &\quad + \mathcal{O}[u, S_2]U + D\mathcal{O}[u, C_1]U - \mathcal{O}[u, C_1]DU + \mathcal{O}[\langle \partial_x \rangle u, C_2]U \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, Q_1]U + \mathcal{O}[u, Q_2]U) \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U) \\ &\quad + \mathcal{O}\left[\frac{\partial_x}{\langle \partial_x \rangle}((\langle \partial_x \rangle u)^2 + (r_x)^2), A_1\right]U + \mathcal{O}[2\langle \partial_x \rangle u r_x, A_2]U \\ &\quad + \mathcal{O}\left[\frac{\partial_x}{\langle \partial_x \rangle}((\langle \partial_x \rangle u)^2 + (r_x)^2), C_1\right]U + \mathcal{O}[2\langle \partial_x \rangle u r_x, C_2]U \\ &= \mathcal{O}[r, Q_1 - B_1]U + \mathcal{O}[u, Q_2 - B_2]U \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, Q_1]U + \mathcal{O}[u, Q_2]U) \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U) \\ &\quad + I_1 + I_2, \end{aligned}$$

where

$$I_1 := \mathcal{O}\left[\frac{\partial_x}{\langle \partial_x \rangle}((\langle \partial_x \rangle u)^2 + (r_x)^2), A_1\right]U + \mathcal{O}[2\langle \partial_x \rangle u r_x, A_2]U, \quad (2.47)$$

$$I_2 := \mathcal{O}\left[\frac{\partial_x}{\langle \partial_x \rangle}((\langle \partial_x \rangle u)^2 + (r_x)^2), C_1\right]U + \mathcal{O}[2\langle \partial_x \rangle u r_x, C_2]U. \quad (2.48)$$

Now using (2.46), we conclude that

$$\Phi_t + D\Phi - \mathcal{O}[r, Q_1 - B_1]\Phi - \mathcal{O}[u, Q_2 - B_2]\Phi = I_1 + I_2 + I_3 + I_4. \quad (2.49)$$

Here,  $I_1, I_2$  are defined by (2.47), (2.48), respectively, and

$$I_3 := -[\mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2], \mathcal{O}[u, A_1] + \mathcal{O}[r, A_2]]U, \quad (2.50)$$

where the notation  $[\cdot, \cdot]$  denotes the commutator, and

$$\begin{aligned}
I_4 := & -(\mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2])(\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])U \\
& + (\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2])U \\
& + (\mathcal{O}[r, B_1] + \mathcal{O}[u, B_2])(\mathcal{O}[u, A_1]U + \mathcal{O}[r, A_2]U + \mathcal{O}[u, C_1]U + \mathcal{O}[r, C_2]U) \\
& + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U).
\end{aligned} \tag{2.51}$$

## 2.4 Energy estimate

**Proposition 2.7** *Solutions of the equation (2.49) satisfy*

$$\frac{d}{dt}\|\Phi\|_{H^N}^2 \lesssim \|U\|_{C^5}^2 \|U\|_{H^N} \|\Phi\|_{H^N}.$$

Proposition 2.7 will be proved by Lemmas 2.8–2.11 and Lemma 2.15.

**Lemma 2.8** *Let  $A_1$  and  $A_2$  be given by (2.30), then for any  $\rho \geq 4$ ,*

$$\|\langle \partial_x \rangle^N \mathcal{O}[f, A_1]U\|_{L^2} + \|\langle \partial_x \rangle^N \mathcal{O}[f, A_2]U\|_{L^2} \lesssim \|f\|_{C^\rho} \|U\|_{H^N}. \tag{2.52}$$

*Proof.* The proof is similar to (2.17). By the definition of  $\mathcal{O}[u, A_1]U$ , we have

$$\begin{aligned}
\langle \partial_x \rangle^N \mathcal{O}[f, A_1]U &= \frac{1}{(2\pi)^2} \langle \partial_x \rangle^N \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{f}(\xi_1) A_1(\xi_1, \xi_2) \widehat{U}(\xi_2) d\xi_1 d\xi_2 \\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix(\xi_1 + \xi_2)} \widehat{\langle \partial_x \rangle^\rho f}(\xi_1) M(\xi_1, \xi_2) \widehat{\langle \partial_x \rangle^N U}(\xi_2) d\xi_1 d\xi_2,
\end{aligned}$$

where

$$M(\xi_1, \xi_2) = \frac{\langle \xi_1 + \xi_2 \rangle^N}{\langle \xi_1 \rangle^\rho \langle \xi_2 \rangle^N} A_1(\xi_1, \xi_2), \quad |\xi_1| \ll |\xi_2|.$$

Note that (2.30) and (2.32) imply

$$\|M(\xi_1, \xi_2 - \xi_1)\|_{L_{\xi_2}^\infty H_{\xi_1}^1} \lesssim \|\langle \xi_1 \rangle^{3-\rho}\|_{L^2} \lesssim 1, \quad \rho \geq 4.$$

Using Lemma 2.4, we thus obtain

$$\|\langle \partial_x \rangle^N \mathcal{O}[f, A_1]U\|_{L^2} \lesssim \|f\|_{C^\rho} \|U\|_{H^N}.$$

The estimate for  $\|\mathcal{O}[f, A_2]U\|_{H^N}$  is the same. □

As a direct consequence of Lemma 2.8, we have

**Lemma 2.9** *For any  $\alpha \in \mathbb{N}$ , we have*

$$\|\langle \partial_x \rangle^N \mathcal{O}[\langle \partial_x \rangle^\alpha f, A_1]U\|_{L^2} + \|\langle \partial_x \rangle^N \mathcal{O}[\langle \partial_x \rangle^\alpha f, A_2]U\|_{L^2} \lesssim \|f\|_{C^{\alpha+4}} \|U\|_{H^N}. \tag{2.53}$$

Moreover, with the same argument as Lemma 2.8, we can obtain the following lemma.

**Lemma 2.10** Let  $Q_1$  and  $Q_2$  be defined by (2.8)–(2.9), then for any  $\rho \geq 2$ , there holds

$$\|\langle \partial_x \rangle^N \mathcal{O}[f, Q_1]U\|_{L^2} + \|\langle \partial_x \rangle^N \mathcal{O}[f, Q_2]U\|_{L^2} \lesssim \|f\|_{C^\rho} \|U\|_{H^{N+1}}. \quad (2.54)$$

According to (2.3), the support of  $1 - \theta(\xi_1, \xi_2) - \theta(\xi_2, \xi_1)$  satisfies  $|\xi_1| \sim |\xi_2|$ . Therefore, we have the following results called “derivative sharing” lemma.

**Lemma 2.11** For any  $\rho, \mu \in \mathbb{N}$ ,  $\rho + \mu = N + 2$ , then

$$\|\langle \partial_x \rangle^N \mathcal{O}[f, C_1]U\|_{L^2} + \|\langle \partial_x \rangle^N \mathcal{O}[f, C_2]U\|_{L^2} \lesssim \|f\|_{C^{\rho+2}} \|U\|_{H^\mu}, \quad (2.55)$$

$$\|\langle \partial_x \rangle^N \mathcal{O}[f, S_1]U\|_{L^2} + \|\langle \partial_x \rangle^N \mathcal{O}[f, S_2]U\|_{L^2} \lesssim \|f\|_{C^{\rho+2}} \|U\|_{H^\mu}, \quad (2.56)$$

where  $C_1, C_2$  and  $S_1, S_2$  are defined by (2.43) and (2.10)–(2.11), respectively.

*Proof.* For  $j = 1, 2$ , we have

$$\begin{aligned} \langle \partial_x \rangle^N \mathcal{O}[f, C_j]U &= \frac{1}{(2\pi)^2} \langle \partial_x \rangle^N \int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)} \widehat{f}(\xi_1) C_j(\xi_1, \xi_2) \widehat{U}(\xi_2) d\xi_1 d\xi_2 \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix(\xi_1+\xi_2)} \langle \widehat{\partial_x}^{\rho+2} f(\xi_1) \rangle^N \frac{\langle \xi_1 + \xi_2 \rangle^N C_j(\xi_1, \xi_2)}{\langle \xi_1 \rangle^{\rho+2} \langle \xi_2 \rangle^\mu} \langle \widehat{\partial_x}^\mu U(\xi_2) \rangle d\xi_1 d\xi_2. \end{aligned}$$

Let

$$M_j(\xi_1, \xi_2) = \frac{\langle \xi_1 + \xi_2 \rangle^N C_j(\xi_1, \xi_2)}{\langle \xi_1 \rangle^{\rho+2} \langle \xi_2 \rangle^\mu}, \quad |\xi_1| \sim |\xi_2|, \quad j = 1, 2,$$

from (2.43) and (2.45), it is easy to see

$$\|M_j(\xi_1, \xi_2 - \xi_1)\|_{L_{\xi_2}^\infty H_{\xi_1}^1} = \left\| \frac{\langle \xi_1 \rangle^N C_j(\xi_1, \xi_2 - \xi_1)}{\langle \xi_1 \rangle^{\rho+2} \langle \xi_2 - \xi_1 \rangle^\mu} \right\|_{L_{\xi_2}^\infty H_{\xi_1}^1} \lesssim 1, \quad j = 1, 2. \quad (2.57)$$

Hence, the desired estimate (2.55) follows from (2.57) and Lemma 2.4. The proof for (2.56) is similar as above.  $\square$

**Lemma 2.12** Let

$$\mathcal{O}[f_1, f_2, M]V(x) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\xi+\eta+\sigma)} \widehat{f}_1(\xi) \widehat{f}_2(\eta) M(\xi, \eta, \sigma) \widehat{V}(\sigma) d\xi d\eta d\sigma,$$

then we have

$$\|\mathcal{O}[f_1, f_2, M]V\|_{L^2(\mathbb{R})} \lesssim \|M(\xi, \eta - \xi, \sigma - \eta)\|_{L_\sigma^\infty H_\xi^1 H_\eta^1} \|f_1\|_{L^\infty} \|f_2\|_{L^\infty} \|V\|_{L^2}.$$

This lemma can be proved by applying similar argument as the proof of Lemma 2.4. The following two lemmas are crucial in proving Lemma 2.15 below.

**Lemma 2.13** Assume  $|\xi_1|, |\eta| \ll |\xi_2|$ , then for any  $\alpha, \beta, \gamma = 0, 1$ , we have

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (q_1(\xi_1, \xi_2) + q_4(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle, \quad (2.58)$$

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (q_2(\xi_1, \xi_2) - q_3(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle, \quad (2.59)$$

$$|\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma (q_j(\eta, \xi_1 + \xi_2) - q_j(\eta, \xi_2))| \lesssim \langle \eta \rangle \langle \xi_1 \rangle, \quad j = 1, 2, 3, 4. \quad (2.60)$$

From the definitions (2.8)–(2.9), one can easily obtain the bounds (2.58)–(2.60).

**Lemma 2.14** *Assume  $|\xi_1|, |\eta| \ll |\xi_2|$ , then for any  $\alpha, \beta, \gamma = 0, 1$ , we have*

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (a_1(\xi_1, \xi_2) + a_4(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle^4 \langle \xi_2 \rangle^{-1}, \quad (2.61)$$

$$|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (a_2(\xi_1, \xi_2) - a_3(\xi_1, \xi_2))| \lesssim \langle \xi_1 \rangle^4 \langle \xi_2 \rangle^{-1}, \quad (2.62)$$

$$|\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma (a_j(\xi_1, \eta + \xi_2) - a_j(\xi_1, \xi_2))| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^3 \langle \xi_2 \rangle^{-1}, \quad j = 1, 2, 3, 4. \quad (2.63)$$

*Proof.* It follows from (2.34) that

$$\begin{aligned} & a_1(\xi_1, \xi_2) + a_4(\xi_1, \xi_2) \\ &= \frac{1}{G} (-\langle \xi_1 \rangle^2 + \langle \xi_2 \rangle^2 + \langle \xi_1 + \xi_2 \rangle^2 - 2\langle \xi_2 \rangle \langle \xi_1 + \xi_2 \rangle) \cdot (\langle \xi_1 \rangle b_1 - \langle \xi_2 \rangle b_2 + \langle \xi_1 + \xi_2 \rangle b_3) \\ &\quad + \frac{1}{G} (-\langle \xi_1 \rangle^2 + \langle \xi_2 \rangle^2 + \langle \xi_1 + \xi_2 \rangle^2 - 2\langle \xi_2 \rangle \langle \xi_1 + \xi_2 \rangle) \cdot (\langle \xi_1 \rangle b_4 + \langle \xi_2 \rangle b_3 - \langle \xi_1 + \xi_2 \rangle b_2). \end{aligned}$$

Recall that  $G = 2\xi_1^2 + 2\xi_2^2 + 2(\xi_1 + \xi_2)^2 + 3$ . Now, using (2.35)–(2.37), we can obtain (2.61) as desired. The proof of (2.62) is similar, so we skip it. By observing the structure of the expressions for  $a_i$ , we see that in order to prove (2.63), it suffices to show

$$\begin{aligned} & |\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta (b_j(\xi_1, \eta + \xi_2) - b_j(\xi_1, \xi_2))| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^2 \langle \xi_2 \rangle^{-1}, \quad j = 1, 2, 3, 4, \\ & |\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma [\langle \xi_1 + \xi_2 + \eta \rangle b_3(\xi_1, \xi_2 + \eta) - \langle \xi_2 + \eta \rangle b_2(\xi_1, \xi_2 + \eta) \\ &\quad - \langle \xi_1 + \xi_2 \rangle b_3(\xi_1, \xi_2) + \langle \xi_2 \rangle b_2(\xi_1, \xi_2)]| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^3 \langle \xi_2 \rangle^{-1}, \\ & |\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma [\langle \xi_2 + \eta \rangle b_3(\xi_1, \xi_2 + \eta) - \langle \xi_1 + \xi_2 + \eta \rangle b_2(\xi_1, \xi_2 + \eta) \\ &\quad - \langle \xi_2 \rangle b_3(\xi_1, \xi_2) + \langle \xi_1 + \xi_2 \rangle b_2(\xi_1, \xi_2)]| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^3 \langle \xi_2 \rangle^{-1}, \end{aligned}$$

and

$$\begin{aligned} & |\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma [\langle \xi_1 + \xi_2 + \eta \rangle b_4(\xi_1, \xi_2 + \eta) + \langle \xi_2 + \eta \rangle b_1(\xi_1, \xi_2 + \eta) \\ &\quad - \langle \xi_1 + \xi_2 \rangle b_4(\xi_1, \xi_2) - \langle \xi_2 \rangle b_1(\xi_1, \xi_2)]| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^3 \langle \xi_2 \rangle^{-1}, \\ & |\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma [\langle \xi_2 + \eta \rangle b_4(\xi_1, \xi_2 + \eta) + \langle \xi_1 + \xi_2 + \eta \rangle b_1(\xi_1, \xi_2 + \eta) \\ &\quad - \langle \xi_2 \rangle b_4(\xi_1, \xi_2) - \langle \xi_1 + \xi_2 \rangle b_1(\xi_1, \xi_2)]| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^3 \langle \xi_2 \rangle^{-1}. \end{aligned}$$

These estimates follow by (2.22), (2.27) and an elementary but tedious computation. We omit the details for simplicity.  $\square$

**Lemma 2.15** *The following four commutator estimates hold:*

$$\|[\mathcal{O}[r, Q_1], \mathcal{O}[u, A_1]]U\|_{H^N} \lesssim \|r\|_{C^5} \|u\|_{C^5} \|U\|_{H^N}, \quad (2.64)$$

$$\|[\mathcal{O}[r, Q_1], \mathcal{O}[r, A_2]]U\|_{H^N} \lesssim \|r\|_{C^5}^2 \|U\|_{H^N}, \quad (2.65)$$

$$\|[\mathcal{O}[u, Q_2], \mathcal{O}[u, A_1]]U\|_{H^N} \lesssim \|u\|_{C^5}^2 \|U\|_{H^N}, \quad (2.66)$$

$$\|[\mathcal{O}[u, Q_2], \mathcal{O}[r, A_2]]U\|_{H^N} \lesssim \|r\|_{C^5} \|u\|_{C^5} \|U\|_{H^N}. \quad (2.67)$$

*Proof.* We first show (2.64). Note that

$$\begin{aligned} \mathcal{O}[r, Q_1](\mathcal{O}[u, A_1]U) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\eta+\xi_2)} \widehat{r}(\eta) Q_1(\eta, \xi_2) \widehat{u}(\xi_1) A_1(\xi_1, \xi_2 - \xi_1) \widehat{U}(\xi_2 - \xi_1) d\eta d\xi_1 d\xi_2 \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\eta+\xi_1+\xi_2)} \widehat{r}(\eta) \widehat{u}(\xi_1) Q_1(\eta, \xi_1 + \xi_2) A_1(\xi_1, \xi_2) \widehat{U}(\xi_2) d\eta d\xi_1 d\xi_2, \end{aligned}$$

and

$$\begin{aligned} \mathcal{O}[u, A_1](\mathcal{O}[r, Q_1]U) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\xi_1+\xi_2)} \widehat{u}(\xi_1) A_1(\xi_1, \xi_2) \widehat{r}(\eta) Q_1(\eta, \xi_2 - \eta) \widehat{U}(\xi_2 - \eta) d\eta d\xi_1 d\xi_2 \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\eta+\xi_1+\xi_2)} \widehat{r}(\eta) \widehat{u}(\xi_1) A_1(\xi_1, \xi_2 + \eta) Q_1(\eta, \xi_2) \widehat{U}(\xi_2) d\eta d\xi_1 d\xi_2, \end{aligned}$$

Hence, we obtain

$$[\mathcal{O}[r, Q_1], \mathcal{O}[u, A_1]]U = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\eta+\xi_1+\xi_2)} \widehat{r}(\eta) \widehat{u}(\xi_1) M_1(\eta, \xi_1, \xi_2) \widehat{U}(\xi_2) d\eta d\xi_1 d\xi_2,$$

where

$$M_1(\eta, \xi_1, \xi_2) := Q_1(\eta, \xi_1 + \xi_2) A_1(\xi_1, \xi_2) - A_1(\xi_1, \xi_2 + \eta) Q_1(\eta, \xi_2). \quad (2.68)$$

From the support property of  $Q_1$  and  $A_1$ , we know that the support of  $M(\eta, \xi_1, \xi_2)$  satisfies  $|\eta|, |\xi_1| \ll |\xi_2|$ . Using this fact, in order to prove (2.64), it suffices to show

$$|\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma M_1(\eta, \xi_1, \xi_2)| \lesssim \langle \xi_1 \rangle^4 \langle \eta \rangle^4, \quad \alpha, \beta, \gamma = 0, 1. \quad (2.69)$$

Indeed, if (2.69) holds, we have

$$|\partial_\eta^\alpha \partial_{\xi_1}^\beta M_1(\eta, \xi_1 - \eta, \xi_2 - \xi_1)| \lesssim \langle \xi_1 - \eta \rangle^4 \langle \eta \rangle^4, \quad \alpha, \beta = 0, 1,$$

then according to Lemma 2.12, the estimate (2.64) thus follows.

To prove (2.69), we decompose the symbol  $M_1$  into  $M_{11} + M_{12} + M_{13}$  with

$$\begin{aligned} M_{11}(\eta, \xi_1, \xi_2) &:= (Q_1(\eta, \xi_1 + \xi_2) - Q_1(\eta, \xi_2)) A_1(\xi_1, \xi_2), \\ M_{12}(\eta, \xi_1, \xi_2) &:= (A_1(\xi_1, \xi_2) - A_1(\xi_1, \xi_2 + \eta)) Q_1(\eta, \xi_2), \\ M_{13}(\eta, \xi_1, \xi_2) &:= Q_1(\eta, \xi_2) A_1(\xi_1, \xi_2) - A_1(\xi_1, \xi_2) Q_1(\eta, \xi_2) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot (q_1(\eta, \xi_2) a_4(\xi_1, \xi_2) - q_4(\eta, \xi_2) a_1(\xi_1, \xi_2)). \end{aligned}$$

For the symbol  $M_{11}$ , we use (2.32) and (2.60), then

$$|\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma M_{11}(\eta, \xi_1, \xi_2)| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^4.$$

For  $M_{12}$ , by (2.8) and (2.63), it is easy to see

$$|\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma M_{12}(\eta, \xi_1, \xi_2)| \lesssim \langle \eta \rangle^2 \langle \xi_1 \rangle^3.$$

For the last symbol  $M_{13}$ , we note that

$$\begin{aligned} q_1(\eta, \xi_2)a_4(\xi_1, \xi_2) - q_4(\eta, \xi_2)a_1(\xi_1, \xi_2) &= (q_1(\eta, \xi_2) + q_4(\eta, \xi_2))a_4(\xi_1, \xi_2) \\ &\quad - q_4(\eta, \xi_2)(a_1(\xi_1, \xi_2) + a_4(\xi_1, \xi_2)), \end{aligned}$$

then it can be inferred from (2.58), (2.32), (2.61) and (2.8) that

$$|\partial_\eta^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma M_{13}(\eta, \xi_1, \xi_2)| \lesssim \langle \eta \rangle \langle \xi_1 \rangle^4.$$

Combing the above three bounds yield (2.69). This finishes the proof of (2.64).

We then turn to show (2.65)–(2.67). Notice that

$$\begin{aligned} [\mathcal{O}[r, Q_1], \mathcal{O}[r, A_2]]U &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\eta+\xi_1+\xi_2)} \widehat{r}(\eta) \widehat{r}(\xi_1) M_2(\eta, \xi_1, \xi_2) \widehat{U}(\xi_2) d\eta d\xi_1 d\xi_2, \\ [\mathcal{O}[u, Q_2], \mathcal{O}[u, A_1]]U &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\eta+\xi_1+\xi_2)} \widehat{u}(\eta) \widehat{u}(\xi_1) M_3(\eta, \xi_1, \xi_2) \widehat{U}(\xi_2) d\eta d\xi_1 d\xi_2, \\ [\mathcal{O}[u, Q_2], \mathcal{O}[r, A_2]]U &:= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix(\eta+\xi_1+\xi_2)} \widehat{u}(\eta) \widehat{r}(\xi_1) M_4(\eta, \xi_1, \xi_2) \widehat{U}(\xi_2) d\eta d\xi_1 d\xi_2, \end{aligned}$$

where

$$\begin{aligned} M_2(\eta, \xi_1, \xi_2) &:= Q_1(\eta, \xi_1 + \xi_2) A_2(\xi_1, \xi_2) - A_2(\xi_1, \xi_2 + \eta) Q_1(\eta, \xi_2), \\ M_3(\eta, \xi_1, \xi_2) &:= Q_2(\eta, \xi_1 + \xi_2) A_1(\xi_1, \xi_2) - A_1(\xi_1, \xi_2 + \eta) Q_2(\eta, \xi_2), \\ M_4(\eta, \xi_1, \xi_2) &= Q_2(\eta, \xi_1 + \xi_2) A_2(\xi_1, \xi_2) - A_2(\xi_1, \xi_2 + \eta) Q_2(\eta, \xi_2). \end{aligned}$$

Applying Lemma 2.12 and repeating similar argument as proof of (2.69), (2.65)–(2.67) can be proved as desired. Since the proof is very similar to the symbol (2.68), we omit further details.  $\square$

*Proof of Proposition 2.7.* Performing energy estimate at  $H^N$  level for (2.49), we have

$$\begin{aligned} \operatorname{Re}\langle \langle \partial_x \rangle^N \Phi_t + \langle \partial_x \rangle^N D\Phi, \langle \partial_x \rangle^N \Phi \rangle - \operatorname{Re}\langle \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1] \Phi, \langle \partial_x \rangle^N \Phi \rangle \\ - \operatorname{Re}\langle \langle \partial_x \rangle^N \mathcal{O}[u, Q_2 - B_2] \Phi, \langle \partial_x \rangle^N \Phi \rangle = \operatorname{Re}\langle \langle \partial_x \rangle^N (I_1 + I_2 + I_3 + I_4), \langle \partial_x \rangle^N \Phi \rangle, \end{aligned} \quad (2.70)$$

where

$$\operatorname{Re}\langle \langle \partial_x \rangle^N \Phi_t, \langle \partial_x \rangle^N \Phi \rangle = \frac{1}{2} \frac{d}{dt} \|\Phi\|_{H^N}^2, \quad \operatorname{Re}\langle \langle \partial_x \rangle^N D\Phi, \langle \partial_x \rangle^N \Phi \rangle = 0, \quad (2.71)$$

and from (2.14)–(2.15),

$$\operatorname{Re}\langle \langle \partial_x \rangle^N \mathcal{O}[r, Q_1 - B_1] \Phi, \langle \partial_x \rangle^N \Phi \rangle = 0, \quad \operatorname{Re}\langle \langle \partial_x \rangle^N \mathcal{O}[u, Q_2 - B_2] \Phi, \langle \partial_x \rangle^N \Phi \rangle = 0. \quad (2.72)$$

It remains to estimate the nonlinear terms in the right hand side of (2.70).

First, we consider  $I_1$ . From (2.47) and (2.53),

$$\begin{aligned}
|\operatorname{Re}\langle\langle\partial_x\rangle^N I_1, \langle\partial_x\rangle^N \Phi\rangle| &\lesssim |\langle\langle\partial_x\rangle^N \mathcal{O}[\frac{\partial_x}{\langle\partial_x\rangle}((\langle\partial_x\rangle u)^2 + (r_x)^2), A_1]U, \langle\partial_x\rangle^N \Phi\rangle| \\
&\quad + |\langle\langle\partial_x\rangle^N \mathcal{O}[2\langle\partial_x\rangle u r_x, A_2]U, \langle\partial_x\rangle^N \Phi\rangle| \\
&\lesssim (\|r^2\|_{C^5} + \|u^2\|_{C^5} + \|ru\|_{C^5})\|U\|_{H^N}\|\Phi\|_{H^N} \\
&\lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2\|U\|_{H^N}\|\Phi\|_{H^N}.
\end{aligned} \tag{2.73}$$

Similarly, from (2.48) and (2.55), there holds

$$|\operatorname{Re}\langle\langle\partial_x\rangle^N I_2, \langle\partial_x\rangle^N \Phi\rangle| \lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2\|U\|_{H^N}\|\Phi\|_{H^N}. \tag{2.74}$$

Next, we consider  $I_4$ . Decompose  $I_4$  (see (2.51)) into  $I_{41} + I_{42} + I_{43} + I_{44} + I_{45}$  with

$$\begin{aligned}
I_{41} &:= -(\mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2])(\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])U, \\
I_{42} &:= (\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2])U, \\
I_{43} &:= (\mathcal{O}[r, B_1] + \mathcal{O}[u, B_2])(\mathcal{O}[u, A_1]U + \mathcal{O}[r, A_2]U), \\
I_{44} &:= (\mathcal{O}[r, B_1] + \mathcal{O}[u, B_2])(\mathcal{O}[u, C_1]U + \mathcal{O}[r, C_2]U), \\
I_{45} &:= (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U).
\end{aligned}$$

From Lemma 2.10 and Lemma 2.11,

$$\begin{aligned}
|\operatorname{Re}\langle\langle\partial_x\rangle^N I_{41}, \langle\partial_x\rangle^N \Phi\rangle| &\lesssim (\|r\|_{C^5} + \|u\|_{C^5})\|(\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])U\|_{H^{N+1}}\|\Phi\|_{H^N}, \\
&\lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2\|U\|_{H^N}\|\Phi\|_{H^N},
\end{aligned} \tag{2.75}$$

$$\begin{aligned}
|\operatorname{Re}\langle\langle\partial_x\rangle^N I_{42}, \langle\partial_x\rangle^N \Phi\rangle| &\lesssim (\|r\|_{C^5} + \|u\|_{C^5})\|(\mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2])U\|_{H^{N-1}}\|\Phi\|_{H^N}, \\
&\lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2\|U\|_{H^N}\|\Phi\|_{H^N}.
\end{aligned} \tag{2.76}$$

From (2.17) and Lemma 2.8, we have

$$\begin{aligned}
|\operatorname{Re}\langle\langle\partial_x\rangle^N I_{43}, \langle\partial_x\rangle^N \Phi\rangle| &\lesssim \|(\mathcal{O}[r, B_1] + \mathcal{O}[u, B_2])(\mathcal{O}[u, A_1]U + \mathcal{O}[r, A_2]U)\|_{H^N}\|\Phi\|_{H^N} \\
&\lesssim (\|r\|_{C^5} + \|u\|_{C^5})\|\mathcal{O}[u, A_1]U + \mathcal{O}[r, A_2]U\|_{H^N}\|\Phi\|_{H^5} \\
&\lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2\|U\|_{H^N}\|\Phi\|_{H^N}.
\end{aligned} \tag{2.77}$$

For the terms  $I_{44}$  and  $I_{45}$ , we use (2.17), Lemma 2.8 and Lemma 2.11 to obtain

$$\begin{aligned}
|\operatorname{Re}\langle\langle\partial_x\rangle^N I_{44}, \langle\partial_x\rangle^N \Phi\rangle| &\lesssim (\|r\|_{C^5} + \|u\|_{C^5})\|\mathcal{O}[u, C_1]U + \mathcal{O}[r, C_2]U\|_{H^N}\|\Phi\|_{H^N} \\
&\lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2\|U\|_{H^N}\|\Phi\|_{H^N},
\end{aligned} \tag{2.78}$$

$$\begin{aligned}
|\operatorname{Re}\langle\langle\partial_x\rangle^N I_{45}, \langle\partial_x\rangle^N \Phi\rangle| &\lesssim (\|r\|_{C^5} + \|u\|_{C^5})\|\mathcal{O}[r, S_1]U + \mathcal{O}[u, S_2]U\|_{H^N}\|\Phi\|_{H^N} \\
&\lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2\|U\|_{H^N}\|\Phi\|_{H^N}.
\end{aligned} \tag{2.79}$$

At last, we consider the term  $I_3$  (see (2.50)), which is a commutator operator. Indeed, applying Lemma 2.15, we see

$$|\operatorname{Re}\langle \langle \partial_x \rangle^N I_3, \langle \partial_x \rangle^N \Phi \rangle| \lesssim (\|r\|_{C^5} + \|u\|_{C^5})^2 \|U\|_{H^N} \|\Phi\|_{H^N}. \quad (2.80)$$

Now, combining the estimates (2.70)–(2.80), we obtain

$$\frac{d}{dt} \|\Phi\|_{H^N}^2 \lesssim \|U\|_{C^5}^2 \|U\|_{H^N} \|\Phi\|_{H^N}.$$

This ends the proof of Proposition 2.7.  $\square$

Finally, we prove the energy estimate stated at the beginning of this section.

*Proof of Proposition 2.1.* It follows from (2.46), Lemma 2.8 and Lemma 2.11 that

$$\|\Phi(t)\|_{H^N} \lesssim \|U(t)\|_{H^N} + \|U(t)\|_{C^5} \|U(t)\|_{H^N},$$

and equally

$$\|U(t)\|_{H^N} \lesssim \|\Phi(t)\|_{H^N} + \|U(t)\|_{C^5} \|\Phi(t)\|_{H^N}.$$

Using (2.1), we notice that if  $\epsilon_1$  is sufficiently small, then

$$\|U(t)\|_{H^N} \lesssim \|\Phi(t)\|_{H^N} \lesssim \|U(t)\|_{H^N}.$$

Hence, Proposition 2.7 and the a-priori bound (2.1) yield

$$\frac{d}{dt} \|\Phi\|_{H^N}^2 \lesssim \|U\|_{C^5}^2 \|U\|_{H^N}^2 \lesssim \epsilon_1^4 (1+t)^{2p_0-1}.$$

Integrating this estimate and using (1.7), we deduce the desired bound (2.2).  $\square$

### 3 Low energy estimate for $\Gamma U$ and $xU$

In this section, we will prove the following two propositions, which lead to the low energy estimate ( $H^{N_1}$  norm) of  $\Gamma U = (x\partial_t + t\partial_x)U$  and  $xU$ , where  $N_1 \ll N$ .

**Proposition 3.1** *Let  $U(t) \in C([0, T]; H^N)$  be the solution of system (1.6). Assume (1.7) holds and*

$$\begin{aligned} \sup_{t \in [0, T]} [(1+t)^{-p_0} \|U(t)\|_{H^N} + (1+t)^{-p_0} \|\Gamma U(t)\|_{H^{N_1}} \\ + (1+t)^{1/2} \|U(t)\|_{W^{N_1+10, \infty}}] \lesssim \epsilon_1, \end{aligned} \quad (3.1)$$

where  $N_1 = 15$ ,  $N = 300$ ,  $0 < p_0 < 10^{-3}$  and  $0 < \epsilon_0 \ll \epsilon_1 \ll 1$ , then

$$\sup_{t \in [0, T]} [(1+t)^{-p_0} \|\Gamma U(t)\|_{H^{N_1}}] \lesssim \epsilon_0 + \epsilon_1^2. \quad (3.2)$$

**Proposition 3.2** *Under the same assumptions as Proposition 3.1, we have*

$$\sup_{t \in [0, T]} \|xU(t)\|_{H^{N_1}} \lesssim \epsilon_1(1+t)^{1+p_0}. \quad (3.3)$$

In the following, Sections 3.1-3.3 are devoted to proving Proposition 3.1, and Proposition 3.2 is proved in Section 3.4.

### 3.1 Shatah's normal form for quadratic terms without loss of derivatives

To prove Proposition 3.1, we have to derive the equation for  $\Gamma U$ . Recall the Euler-Poisson system

$$\begin{pmatrix} r \\ u \end{pmatrix}_t + \begin{pmatrix} 0 & -\langle \partial_x \rangle \\ \langle \partial_x \rangle & 0 \end{pmatrix} \begin{pmatrix} r \\ u \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} 2\langle \partial_x \rangle u & r_x \\ \frac{\partial_x}{\langle \partial_x \rangle}[(\langle \partial_x \rangle u)^2 + (r_x)^2] & \end{pmatrix}. \quad (3.4)$$

For simplicity, we write the above system as

$$U_t + DU = (f_1, f_2)^T. \quad (3.5)$$

Operating  $\Gamma$  on both sides of the system (3.4), then using the relations

$$[\Gamma, \partial_x] = -\partial_t, \quad [\Gamma, \partial_t] = -\partial_x, \quad [\Gamma, \langle \partial_x \rangle] = \frac{\partial_x}{\langle \partial_x \rangle} \partial_t, \quad [\Gamma, \frac{\partial_x}{\langle \partial_x \rangle}] = -\frac{1}{\langle \partial_x \rangle^3} \partial_t,$$

we obtain equations for  $\Gamma r, \Gamma u$

$$\begin{pmatrix} \Gamma r \\ \Gamma u \end{pmatrix}_t + \begin{pmatrix} 0 & -\langle \partial_x \rangle \\ \langle \partial_x \rangle & 0 \end{pmatrix} \begin{pmatrix} \Gamma r \\ \Gamma u \end{pmatrix} = \begin{pmatrix} 2\langle \partial_x \rangle (\Gamma u) & r_x + 2\langle \partial_x \rangle u (\Gamma r)_x \\ \frac{2\partial_x}{\langle \partial_x \rangle}[(\partial_x)(\Gamma u)\langle \partial_x \rangle u + (\Gamma r)_x r_x] & \end{pmatrix} + \begin{pmatrix} g'_1 \\ g'_2 \end{pmatrix} + \begin{pmatrix} g''_1 \\ g''_2 \end{pmatrix},$$

where  $g'_1 = g'_1(r, u)$ ,  $g'_2 = g'_2(r, u)$  are quadratic terms without containing  $\Gamma r$  and  $\Gamma u$ ,

$$\begin{aligned} g'_1(r, u) &:= -2(r_x)^2 - 2(\langle \partial_x \rangle u)^2 + \frac{\partial_x^2}{\langle \partial_x \rangle^2}[(r_x)^2 + (\langle \partial_x \rangle u)^2], \\ g'_2(r, u) &:= -\frac{6\partial_x}{\langle \partial_x \rangle}(r_x \langle \partial_x \rangle u) + \frac{2}{\langle \partial_x \rangle^3}(\langle \partial_x \rangle^2 r \langle \partial_x \rangle u) - \frac{2}{\langle \partial_x \rangle^3}(r_x \langle \partial_x \rangle u_x), \end{aligned}$$

and  $g''_1 = g''_1(r, u)$ ,  $g''_2 = g''_2(r, u)$  are cubic terms,

$$\begin{aligned} g''_1(r, u) &:= -4r_x(\langle \partial_x \rangle u)^2 + 2r_x \frac{\partial_x^2}{\langle \partial_x \rangle^2}[(r_x)^2 + (\langle \partial_x \rangle u)^2], \\ g''_2(r, u) &:= -\frac{4\partial_x}{\langle \partial_x \rangle}[(r_x)^2 \langle \partial_x \rangle u] + \frac{2\partial_x}{\langle \partial_x \rangle} \left[ \frac{\partial_x^2}{\langle \partial_x \rangle^2}[(r_x)^2 + (\langle \partial_x \rangle u)^2] \langle \partial_x \rangle u \right] \\ &\quad - \frac{4}{\langle \partial_x \rangle^3}[(r_x \langle \partial_x \rangle u)_x r_x] - \frac{4}{\langle \partial_x \rangle^3}[(r_x r_{xx} + \langle \partial_x \rangle u \langle \partial_x \rangle u_x) \langle \partial_x \rangle u]. \end{aligned}$$

Since our aim is to estimate  $\|\Gamma U\|_{H^{N_1}}$  (recall that  $N_1 \ll N$ ), we simply decompose the quadratic terms  $g'_1, g'_2$  into

$$(g'_1, g'_2)^T = \mathcal{O}[u, \begin{pmatrix} 0 & \tilde{q}_1(\xi_1, \xi_2) \\ \tilde{q}_4(\xi_1, \xi_2) & 0 \end{pmatrix}]U + \mathcal{O}[r, \begin{pmatrix} \tilde{q}_2(\xi_1, \xi_2) & 0 \\ 0 & \tilde{q}_3(\xi_1, \xi_2) \end{pmatrix}]U$$

with

$$\begin{aligned} \tilde{q}_1(\xi_1, \xi_2) &:= -2\langle \xi_1 \rangle \langle \xi_2 \rangle - \frac{(\xi_1 + \xi_2)^2}{\langle \xi_1 + \xi_2 \rangle^2} \langle \xi_1 \rangle \langle \xi_2 \rangle, \\ \tilde{q}_4(\xi_1, \xi_2) &:= \frac{3(\xi_1 + \xi_2)\langle \xi_1 \rangle \xi_2}{\langle \xi_1 + \xi_2 \rangle} + \frac{\langle \xi_1 \rangle \langle \xi_2 \rangle^2}{\langle \xi_1 + \xi_2 \rangle^3} + \frac{\xi_1 \langle \xi_1 \rangle \xi_2}{\langle \xi_1 + \xi_2 \rangle^3}, \\ \tilde{q}_2(\xi_1, \xi_2) &:= 2\xi_1 \xi_2 + \frac{(\xi_1 + \xi_2)^2}{\langle \xi_1 + \xi_2 \rangle^2} \xi_1 \xi_2, \\ \tilde{q}_3(\xi_1, \xi_2) &:= \frac{3(\xi_1 + \xi_2)\xi_1 \langle \xi_2 \rangle}{\langle \xi_1 + \xi_2 \rangle} + \frac{\langle \xi_1 \rangle^2 \langle \xi_2 \rangle}{\langle \xi_1 + \xi_2 \rangle^3} + \frac{\xi_1 \xi_2 \langle \xi_2 \rangle}{\langle \xi_1 + \xi_2 \rangle^3}. \end{aligned}$$

For the terms including  $\Gamma r$  and  $\Gamma u$ , we use similar decomposition as in Section 2.1,

$$\begin{pmatrix} 2\langle \partial_x \rangle (\Gamma u) r_x + 2\langle \partial_x \rangle u (\Gamma r)_x \\ \frac{2\partial_x}{\langle \partial_x \rangle} [\langle \partial_x \rangle (\Gamma u) \langle \partial_x \rangle u + (\Gamma r)_x r_x] \end{pmatrix} = \begin{pmatrix} F'_1 \\ F'_2 \end{pmatrix} + \begin{pmatrix} F''_1 \\ F''_2 \end{pmatrix}$$

with  $F'_j = F_j(\Gamma r, \Gamma u, r, u)$  and  $F''_j = F'_j(\Gamma r, \Gamma u, r, u)$  ( $j = 1, 2$ ) defined by

$$(F'_1, F'_2)^T := \mathcal{O}(r, Q_1)\Gamma U + \mathcal{O}(u, Q_2)\Gamma U + \mathcal{O}(r, S_1)\Gamma U + \mathcal{O}(u, S_2)\Gamma U, \quad (3.6)$$

$$(F''_1, F''_2)^T := \mathcal{O}(\Gamma r, Q_1)U + \mathcal{O}(\Gamma u, Q_2)U + \mathcal{O}(\Gamma r, S_1)U + \mathcal{O}(\Gamma u, S_2)U, \quad (3.7)$$

where the matrices  $Q_1, Q_2, S_1$  and  $S_2$  are given in (2.8)–(2.11). In conclusion, we obtain

$$(\Gamma U)_t + D\Gamma U = (F'_1, F'_2)^T + (F''_1, F''_2)^T + (g'_1, g'_2)^T + (g''_1, g''_2)^T. \quad (3.8)$$

From (3.7), we see the quadratic terms  $F''_1(\Gamma r, \Gamma u, r, u)$  and  $F''_2(\Gamma r, \Gamma u, r, u)$  will not lead to loss of derivatives when taking the  $H^{N_1}$  norm energy estimate, since  $\Gamma r$  and  $\Gamma u$  have lower frequencies compared to  $U$ . Notice also that the quadratic terms  $g'_1(r, u)$  and  $g'_2(r, u)$  do not contain  $\Gamma r$  and  $\Gamma u$ . For these reasons, we only need to take Shatah's normal form transformation

$$\tilde{\Omega} := \Gamma U + \mathcal{O}[r, G_1]U + \mathcal{O}[u, G_2]U + \mathcal{O}[\Gamma u, H_1]U + \mathcal{O}[\Gamma r, H_2]U \quad (3.9)$$

for the system (3.8) to cancel  $g'_1(r, u)$ ,  $g'_2(r, u)$ ,  $F''_1(\Gamma r, \Gamma u, r, u)$  and  $F''_2(\Gamma r, \Gamma u, r, u)$ . Similar to (2.31), the matrices  $G_1, G_2$  could be obtained from the equations

$$\begin{aligned} -(g'_1, g'_2)^T &= D\mathcal{O}[r, G_1]U - \mathcal{O}[\langle \partial_x \rangle r, G_2]U - \mathcal{O}[r, G_1]DU \\ &\quad + D\mathcal{O}[u, G_2]U + \mathcal{O}[\langle \partial_x \rangle u, G_1]U - \mathcal{O}[u, G_2]DU, \end{aligned} \quad (3.10)$$

and  $H_1, H_2$  can be determined by

$$\begin{aligned} -(F_1'', F_2'')^T &= D\mathcal{O}[\Gamma u, H_1]U + \mathcal{O}[\langle \partial_x \rangle \Gamma u, H_2]U - \mathcal{O}[\Gamma u, H_1]DU \\ &\quad + D\mathcal{O}[\Gamma r, H_2]U - \mathcal{O}[\langle \partial_x \rangle \Gamma r, H_1]U - \mathcal{O}[\Gamma r, H_2]DU. \end{aligned} \quad (3.11)$$

Indeed, the elements of  $G_1, G_2$  (or  $H_1, H_2$ ) satisfy similar linear equations as (2.33), which can be uniquely solved as (2.34). Now using (3.4), (3.5) and (3.8)–(3.11), we reduce (3.8) to

$$\tilde{\Omega}_t + D\tilde{\Omega} = (F_1', F_2')^T + (g_1, g_2)^T, \quad (3.12)$$

where  $(F_1', F_2')^T$  is given as (3.6), and  $g_1(r, u, \Gamma r, \Gamma u), g_2(r, u, \Gamma r, \Gamma u)$  are cubic terms taking the following form

$$\begin{aligned} (g_1, g_2)^T &:= (\mathcal{O}[f_1, G_1] + \mathcal{O}[f_2, G_2])U + (\mathcal{O}[r, G_1] + \mathcal{O}[u, G_2])(f_1, f_2)^T \\ &\quad + \mathcal{O}[F_2' + F_2'' + g_2' + g_2'', H_1]U + \mathcal{O}[F_1' + F_1'' + g_1' + g_1'', H_2]U \\ &\quad + (\mathcal{O}[\Gamma u, H_1] + \mathcal{O}[\Gamma r, H_2])(f_1, f_2)^T + (g_1'', g_2'')^T. \end{aligned} \quad (3.13)$$

Moreover, according to the properties of the symbols  $G_1, G_2, H_1$  and  $H_2$ , we clearly have

$$\begin{aligned} \|\mathcal{O}[f, G_1]V\|_{H^{N_1}} + \|\mathcal{O}[f, G_2]V\|_{H^{N_1}} &\lesssim \|f\|_{C^5}\|V\|_{H^N} + \|f\|_{H^N}\|V\|_{C^5}, \\ \|\mathcal{O}[f, H_1]V\|_{H^{N_1}} + \|\mathcal{O}[f, H_2]V\|_{H^{N_1}} &\lesssim \|f\|_{H^5}\|V\|_{C^{N_1+5}}. \end{aligned} \quad (3.14)$$

### 3.2 Energy normal form for quadratic terms with $\Gamma r$ and $\Gamma u$

Note that the quadratic terms  $F_1'(\Gamma r, \Gamma u, r, u)$  and  $F_2'(\Gamma r, \Gamma u, r, u)$  in (3.12) will lead to loss of derivatives in energy estimate for  $\Gamma U$ , as in these terms  $\Gamma U$  has higher frequencies compared to  $U$ . So in this subsection, we apply similar modified normal form process as in Section 2 to eliminate the derivative quadratic terms  $F_1'$  and  $F_2'$  in (3.12). Taking the energy normal form transformation

$$\Omega := \tilde{\Omega} + \mathcal{O}[u, A_1]\Gamma U + \mathcal{O}[r, A_2]\Gamma U + \mathcal{O}[u, C_1]\Gamma U + \mathcal{O}[r, C_2]\Gamma U, \quad (3.15)$$

where  $A_1, A_2, C_1, C_2$  are completely the same as those defined in (2.30) and (2.43). By repeating similar process as (2.49), we can obtain the equation for  $\Omega$

$$\begin{aligned} \Omega_t + D\Omega &= \mathcal{O}[r, Q_1 - B_1]\Gamma U + \mathcal{O}[u, Q_2 - B_2]\Gamma U + (g_1, g_2)^T \\ &\quad + (\mathcal{O}[f_2, A_1] + \mathcal{O}[f_1, A_2])\Gamma U + (\mathcal{O}[f_2, C_1] + \mathcal{O}[f_1, C_2])\Gamma U \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2])(F_1' + F_1'' + g_1' + g_1'', F_2' + F_2'' + g_2' + g_2'')^T \\ &\quad + (\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(F_1' + F_1'' + g_1' + g_1'', F_2' + F_2'' + g_2' + g_2'')^T, \end{aligned} \quad (3.16)$$

where  $f_1, f_2, F_1', F_2', F_1'', F_2''$  and  $g_1, g_2$  are defined by (3.4), (3.6), (3.7) and (3.13), respectively. From (3.9) and (3.15), we have

$$\begin{aligned} \Omega &= \Gamma U + \mathcal{O}[r, G_1]U + \mathcal{O}[u, G_2]U + \mathcal{O}[\Gamma u, H_1]U + \mathcal{O}[\Gamma r, H_2]U \\ &\quad + \mathcal{O}[u, A_1]\Gamma U + \mathcal{O}[r, A_2]\Gamma U + \mathcal{O}[u, C_1]\Gamma U + \mathcal{O}[r, C_2]\Gamma U. \end{aligned} \quad (3.17)$$

Using (3.17), we rewrite the equation (3.16) as

$$\Omega_t + D\Omega - \mathcal{O}[r, Q_1 - B_1]\Omega - \mathcal{O}[u, Q_2 - B_2]\Omega = J_1 + J_2 + J_3 + J_4 + (g_1, g_2)^T, \quad (3.18)$$

where

$$\begin{aligned} J_1 &:= [\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2], \mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2]]\Gamma U, \\ J_2 &:= -(\mathcal{O}[r, Q_1 - B_1] + \mathcal{O}[u, Q_2 - B_2])(\mathcal{O}[r, G_1]U + \mathcal{O}[u, G_2]U) \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(g'_1 + g''_1, g'_2 + g''_2)^T, \\ J_3 &:= (\mathcal{O}[r, B_1] + \mathcal{O}[u, B_2])(\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])\Gamma U \\ &\quad - (\mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2])(\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])\Gamma U + (\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(F'_1, F'_2)^T \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2])(\mathcal{O}[u, S_1] + \mathcal{O}[r, S_2])\Gamma U \\ &\quad + (\mathcal{O}[f_2, A_1] + \mathcal{O}[f_1, A_2])\Gamma U + (\mathcal{O}[f_2, C_1] + \mathcal{O}[f_1, C_2])\Gamma U, \\ J_4 &:= -(\mathcal{O}[r, Q_1 - B_1] + \mathcal{O}[u, Q_2 - B_2])(\mathcal{O}[\Gamma u, H_1]U + \mathcal{O}[\Gamma r, H_2]U) \\ &\quad + (\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(F''_1, F''_2)^T. \end{aligned}$$

### 3.3 Low energy estimate for $\Gamma U$

*Proof of Proposition 3.1.* The energy estimate for (3.18) is similar to (2.49). Taking energy estimate at  $H^{N_1}$  level for (3.18), we have

$$\begin{aligned} \operatorname{Re}\langle\langle\partial_x\rangle^{N_1}\Omega_t + \langle\partial_x\rangle^{N_1}D\Omega - \langle\partial_x\rangle^{N_1}(\mathcal{O}[r, Q_1 - B_1] - \mathcal{O}[u, Q_2 - B_2])\Omega, \langle\partial_x\rangle^{N_1}\Omega\rangle \\ = \operatorname{Re}\langle\langle\partial_x\rangle^{N_1}(J_1 + J_2 + J_3 + J_4) + \operatorname{Re}\langle\langle\partial_x\rangle^{N_1}(g_1, g_2)^T, \langle\partial_x\rangle^{N_1}\Omega\rangle. \end{aligned}$$

Clearly, there holds

$$\operatorname{Re}\langle\langle\partial_x\rangle^{N_1}\Omega_t, \langle\partial_x\rangle^{N_1}\Omega\rangle = \frac{d}{2dt}\|\Omega\|_{H^{N_1}}^2, \quad \operatorname{Re}\langle\langle\partial_x\rangle^{N_1}D\Omega, \langle\partial_x\rangle^{N_1}\Omega\rangle = 0.$$

Moreover, from (2.14)–(2.15),

$$\operatorname{Re}\langle\langle\partial_x\rangle^{N_1}(\mathcal{O}[r, Q_1 - B_1] + \mathcal{O}[u, Q_2 - B_2])\Omega, \langle\partial_x\rangle^{N_1}\Omega\rangle = 0.$$

For  $J_1$ , we use Lemma 2.15 to obtain

$$|\langle\langle\partial_x\rangle^{N_1}J_1, \langle\partial_x\rangle^{N_1}\Omega\rangle| \lesssim \|U\|_{C^5}^2\|\Gamma U\|_{H^{N_1}}\|\Omega\|_{H^{N_1}}.$$

Note that all the terms in  $J_2$  are cubic terms containing only  $r, u$ , from (2.17), Lemmas 2.8–2.11 and (3.14), we have

$$|\langle\langle\partial_x\rangle^{N_1}J_2, \langle\partial_x\rangle^{N_1}\Omega\rangle| \lesssim \|U\|_{C^5}^2\|U\|_{H^N}\|\Omega\|_{H^{N_1}}.$$

Similar to the estimates for  $I_1, I_2$  and  $I_4$  in (2.49), the term  $J_3$  can be bounded by

$$|\langle\partial_x\rangle^{N_1}J_3, \langle\partial_x\rangle^{N_1}\Omega\rangle| \lesssim \|U\|_{C^5}^2\|\Gamma U\|_{H^{N_1}}\|\Omega\|_{H^{N_1}}.$$

Note that all the terms containing  $\Gamma U$  in  $J_4$  has lower frequencies compared to  $U$ , so

$$|\langle \partial_x \rangle^{N_1} J_4, \langle \partial_x \rangle^{N_1} \Omega \rangle| \lesssim \|U\|_{C^{N_1+9}}^2 \|\Gamma U\|_{H^{N_1}} \|\Omega\|_{H^{N_1}}.$$

Similarly, using (3.14), we obtain

$$|\langle \partial_x \rangle^{N_1} (g_1, g_2)^T, \langle \partial_x \rangle^{N_1} \Omega \rangle| \lesssim \|U\|_{C^{N_1+9}}^2 (\|\Gamma U\|_{H^{N_1}} + \|U\|_{H^N}) \|\Omega\|_{H^{N_1}}.$$

Therefore, we conclude that

$$\frac{d}{dt} \|\Omega\|_{H^{N_1}} \lesssim \|U\|_{C^{N_1+9}}^2 (\|\Gamma U\|_{H^{N_1}} + \|U\|_{H^N}) \lesssim \epsilon_1^3 (1+t)^{p_0-1}. \quad (3.19)$$

where we have used (3.1) in the last step. Note that from (3.14), (3.1) and (2.52)–(2.55), we have

$$\begin{aligned} \|\mathcal{O}[r, G_1]U\|_{H^{N_1}} + \|\mathcal{O}[u, G_2]U\|_{H^{N_1}} &\lesssim \|U\|_{C^5} \|U\|_{H^N} \lesssim \epsilon_1^2 (1+t)^{p_0-1/2} \lesssim \epsilon_1^2, \\ \|\mathcal{O}[\Gamma u, H_1]U\|_{H^{N_1}} + \|\mathcal{O}[\Gamma r, H_2]U\|_{H^{N_1}} &\lesssim \|\Gamma U\|_{H^{N_1}} \|U\|_{C^{N_1+9}} \lesssim \epsilon_1^2 (1+t)^{p_0-1/2} \lesssim \epsilon_1^2, \\ \|\mathcal{O}[u, A_1]\Gamma U\|_{H^{N_1}} + \|\mathcal{O}[r, A_2]\Gamma U\|_{H^{N_1}} &\lesssim \|U\|_{C^5} \|\Gamma U\|_{H^{N_1}} \lesssim \epsilon_1^2 \\ \|\mathcal{O}[u, C_1]\Gamma U\|_{H^{N_1}} + \|\mathcal{O}[r, C_2]\Gamma U\|_{H^{N_1}} &\lesssim \|U\|_{C^5} \|\Gamma U\|_{H^{N_1}} \lesssim \epsilon_1^2. \end{aligned}$$

Hence, we deduce from (3.17) that

$$\|\Omega\|_{H^{N_1}} \lesssim \|\Gamma U\|_{H^{N_1}} + \epsilon_1^2, \quad \|\Gamma U\|_{H^{N_1}} \lesssim \|\Omega\|_{H^{N_1}} + \epsilon_1^2.$$

Integrating (3.19) and using (1.7), we obtain

$$\|\Gamma U\|_{H^{N_1}} \lesssim \epsilon_0 + (1+t)^{p_0} \epsilon_1^2.$$

Proposition 3.1 thus follows.  $\square$

### 3.4 Low energy estimate for $xU$

In this subsection, we aim to prove Proposition 3.2. Using the identities

$$[x, \partial_x] = -I, \quad [x, \partial_t] = 0, \quad [x, \langle \partial_x \rangle] = \frac{\partial_x}{\langle \partial_x \rangle}, \quad [x, \frac{\partial_x}{\langle \partial_x \rangle}] = -\frac{1}{\langle \partial_x \rangle^3},$$

we see  $xr$  and  $xu$  satisfy

$$\begin{pmatrix} xr \\ xu \end{pmatrix}_t + \begin{pmatrix} 0 & -\langle \partial_x \rangle \\ \langle \partial_x \rangle & 0 \end{pmatrix} \begin{pmatrix} xr \\ xu \end{pmatrix} = \begin{pmatrix} (xr)_x \langle \partial_x \rangle u + r_x \langle \partial_x \rangle (xu) \\ \frac{\partial_x}{\langle \partial_x \rangle} [(xr)_x r_x] + \frac{\partial_x}{\langle \partial_x \rangle} [\langle \partial_x \rangle (xu) \langle \partial_x \rangle u] \end{pmatrix} + \begin{pmatrix} N'_1 \\ N'_2 \end{pmatrix}, \quad (3.20)$$

where  $N'_j = N'_j(r, u)$  ( $j = 1, 2$ ) are linear and quadratic terms not including  $xr$  and  $xu$ ,

$$N'_1(r, u) = \frac{\partial_x}{\langle \partial_x \rangle} u - r \langle \partial_x \rangle u + r_x \frac{\partial_x}{\langle \partial_x \rangle} u,$$

$$N'_2(r, u) = -\frac{\partial_x}{\langle \partial_x \rangle} r - \frac{\partial_x}{\langle \partial_x \rangle} (rr_x) + \frac{\partial_x}{\langle \partial_x \rangle} [\frac{\partial_x}{\langle \partial_x \rangle} u (\langle \partial_x \rangle u)] - \frac{1}{\langle \partial_x \rangle^3} (r_x)^2 - \frac{1}{\langle \partial_x \rangle^3} (\langle \partial_x \rangle u)^2.$$

As in (3.6), the first nonlinear term in (3.20) can be decomposed into

$$\begin{pmatrix} (xr)_x \langle \partial_x \rangle u + r_x \langle \partial_x \rangle (xu) \\ \frac{\partial_x}{\langle \partial_x \rangle} [(xr)_x r_x] + \frac{\partial_x}{\langle \partial_x \rangle} [\langle \partial_x \rangle (xu) \langle \partial_x \rangle u] \end{pmatrix} = \frac{1}{2} \begin{pmatrix} F_1(xr, xu, r, u) \\ F_2(xr, xu, r, u) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} F'_1(xr, xu, r, u) \\ F'_2(xr, xu, r, u) \end{pmatrix}$$

with

$$\begin{aligned} (F_1, F_2)^T &= \mathcal{O}[r, Q_1]xU + \mathcal{O}[u, Q_2]xU + \mathcal{O}[r, S_1]xU + \mathcal{O}[u, S_2]xU, \\ (F'_1, F'_2)^T &= \mathcal{O}[xr, Q_1]U + \mathcal{O}[xu, Q_2]U + \mathcal{O}[xr, S_1]U + \mathcal{O}[xu, S_2]U. \end{aligned}$$

Note that  $F_1(xr, xu, r, u)$  and  $F_2(xr, xu, r, u)$  will lead to loss of derivatives for the energy estimate, as  $xU$  has higher frequencies compared to  $r$  and  $u$ . In order to treat this case, we take energy normal form transformation similar to (3.15). Let

$$\begin{aligned} \Theta := xU + \frac{1}{2}(\mathcal{O}[u, A_1]xU + \mathcal{O}[r, A_2]xU + \mathcal{O}[u, C_1]xU + \mathcal{O}[r, C_2]xU) \\ + \frac{1}{2}(\mathcal{O}[xu, H_1]U + \mathcal{O}[xr, H_2]U), \end{aligned} \tag{3.21}$$

where  $H_1$  and  $H_2$  are the same as (3.9), then (3.20) is changed into

$$\begin{aligned} \Theta_t + D\Theta &= \frac{1}{2}\mathcal{O}[r, Q_1 - B_1]xU + \frac{1}{2}\mathcal{O}[u, Q_2 - B_2]xU + (N'_1, N'_2)^T \\ &\quad + \frac{1}{2}(\mathcal{O}[f_2, A_1] + \mathcal{O}[f_1, A_2] + \mathcal{O}[f_2, C_1] + \mathcal{O}[f_1, C_2])xU \\ &\quad + \frac{1}{2}(\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2])(\frac{1}{2}(F_1 + F'_1) + N'_1, \frac{1}{2}(F_2 + F'_2) + N'_2)^T \\ &\quad + \frac{1}{2}(\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(\frac{1}{2}(F_1 + F'_1) + N'_1, \frac{1}{2}(F_2 + F'_2) + N'_2)^T \\ &\quad + \frac{1}{2}(\mathcal{O}[xu, H_1] + \mathcal{O}[xr, H_2])(f_1, f_2)^T \\ &\quad + \frac{1}{2}(\mathcal{O}[\frac{1}{2}(F_2 + F'_2) + N'_2, H_1] + \mathcal{O}[\frac{1}{2}(F_1 + F'_1) + N'_1, H_2])U. \end{aligned}$$

Similar to (3.18), we can obtain

$$\Theta_t + D\Theta - \frac{1}{2}\mathcal{O}[r, Q_1 - B_1]\Theta - \frac{1}{2}[u, Q_2 - B_2]\Theta = L_1 + L_2 + L_3, \tag{3.22}$$

where

$$\begin{aligned} L_1 &:= \frac{1}{4}[\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2], \mathcal{O}[r, Q_1] + \mathcal{O}[u, Q_2]]xU, \\ L_2 &:= (N'_1, N'_2)^T + \frac{1}{2}(\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2] + \mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(N'_1, N'_2)^T \\ &\quad + \frac{1}{2}(\mathcal{O}[N'_2, H_1] + \mathcal{O}[N'_1, H_2])U, \end{aligned}$$

$$\begin{aligned}
L_3 := & -\frac{1}{4}(\mathcal{O}[r, Q_1 - B_1] + \mathcal{O}[u, Q_2 - B_2])(\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])xU \\
& -\frac{1}{4}(\mathcal{O}[r, Q_1 - B_1] + \mathcal{O}[u, Q_2 - B_2])(\mathcal{O}[xu, H_1] + \mathcal{O}[xr, H_2])U \\
& +\frac{1}{4}(\mathcal{O}[r, B_1] + \mathcal{O}[u, B_2])(\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2])xU \\
& +\frac{1}{4}(\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2])(\mathcal{O}[r, S_1] + \mathcal{O}[u, S_2])xU + \frac{1}{4}(\mathcal{O}[u, A_1] + \mathcal{O}[r, A_2])(F'_1, F'_2)^T \\
& +\frac{1}{4}(\mathcal{O}[u, C_1] + \mathcal{O}[r, C_2])(F_1 + F'_1, F_2 + F'_2)^T \\
& +\frac{1}{2}(\mathcal{O}[f_2, A_1] + \mathcal{O}[f_1, A_2] + \mathcal{O}[f_2, C_1] + \mathcal{O}[f_1, C_2])xU \\
& +\frac{1}{2}(\mathcal{O}[xu, H_1] + \mathcal{O}[xr, H_2])(f_1, f_2)^T + \frac{1}{4}(\mathcal{O}[F_2 + F'_2, H_1] + \mathcal{O}[F_1 + F'_1, H_2])U.
\end{aligned}$$

Remember that, there are linear terms in  $N'_1$  and  $N'_2$ . Now, applying similar argument as Section 3.3, we can obtain

$$\begin{aligned}
|\langle \langle \partial_x \rangle^{N_1} L_1, \langle \partial_x \rangle^{N_1} \Theta \rangle| &\lesssim \|U\|_{C^5}^2 \|xU\|_{H^{N_1}} \|\Theta\|_{H^{N_1}}, \\
|\langle \langle \partial_x \rangle^{N_1} L_2, \langle \partial_x \rangle^{N_1} \Theta \rangle| &\lesssim (\|U\|_{H^N} + \|U\|_{C^5} \|U\|_{H^N} + \|U\|_{C^5}^2 \|U\|_{H^N}) \|\Theta\|_{H^{N_1}}, \\
|\langle \langle \partial_x \rangle^{N_1} L_3, \langle \partial_x \rangle^{N_1} \Theta \rangle| &\lesssim \|U\|_{C^{N_1+9}}^2 \|xU\|_{H^{N_1}} \|\Theta\|_{H^{N_1}}.
\end{aligned}$$

Therefore, the  $H^{N_1}$  energy estimate for (3.22) is

$$\begin{aligned}
\frac{d}{dt} \|\Theta\|_{H^{N_1}} &\lesssim \|U\|_{C^{N_1+9}}^2 \|xU\|_{H^{N_1}} + \|U\|_{C^5}^2 \|U\|_{H^N} + \|U\|_{H^N} \\
&\lesssim \epsilon_1^2 (1+t)^{-1} \|xU\|_{H^{N_1}} + \epsilon_1 (1+t)^{p_0}.
\end{aligned}$$

Note that (3.21) implies  $\|xU\|_{H^{N_1}} \sim \|\Theta\|_{H^{N_1}}$  if  $\epsilon_1$  is small enough, so we have

$$\frac{d}{dt} \|\Theta\|_{H^{N_1}} \lesssim \epsilon_1^2 (1+t)^{-1} \|\Theta\|_{H^{N_1}} + \epsilon_1 (1+t)^{p_0}.$$

Using Gronwall's inequality, we obtain

$$\|xU\|_{H^{N_1}} \sim \|\Theta\|_{H^{N_1}} \lesssim \epsilon_0 (1+t)^{C\epsilon_1^2} + \epsilon_1 (1+t)^{1+p_0} \lesssim \epsilon_1 (1+t)^{1+p_0}$$

provided that  $\epsilon_1$  is sufficiently small. This ends the proof of Proposition 3.2.

## 4 Modified scattering and decay estimate

In this section, we will prove Proposition 4.1 below. Recalling (1.8) and (1.9), we have

$$\begin{aligned}
h_t + i\langle \partial_x \rangle h &= \frac{1}{2i}(h + \bar{h})_x \langle \partial_x \rangle (h - \bar{h}) + \frac{\partial_x}{4i\langle \partial_x \rangle} [\langle \partial_x \rangle (h - \bar{h})]^2 - \frac{\partial_x}{4i\langle \partial_x \rangle} [(h + \bar{h})_x]^2 \\
&= \mathcal{O}[h, q^{++}]h + \mathcal{O}[h, q^{+-}]\bar{h} + \mathcal{O}[\bar{h}, q^{--}]\bar{h},
\end{aligned} \tag{4.1}$$

where the expressions for the symbols  $q^{++}$ ,  $q^{+-}$  and  $q^{--}$  are

$$\begin{aligned} q^{++}(\xi, \eta) &:= \frac{1}{2}\xi\langle\eta\rangle + \frac{\xi+\eta}{4\langle\xi+\eta\rangle}\langle\xi\rangle\langle\eta\rangle + \frac{\xi+\eta}{4\langle\xi+\eta\rangle}\xi\eta, \\ q^{+-}(\xi, \eta) &:= -\frac{1}{2}\xi\langle\eta\rangle + \frac{1}{2}\langle\xi\rangle\eta - \frac{\xi+\eta}{2\langle\xi+\eta\rangle}\langle\xi\rangle\langle\eta\rangle + \frac{\xi+\eta}{2\langle\xi+\eta\rangle}\xi\eta, \\ q^{--}(\xi, \eta) &:= -\frac{1}{2}\xi\langle\eta\rangle + \frac{\xi+\eta}{4\langle\xi+\eta\rangle}\langle\xi\rangle\langle\eta\rangle + \frac{\xi+\eta}{4\langle\xi+\eta\rangle}\xi\eta. \end{aligned} \quad (4.2)$$

We first apply Shatah's normal form transformation to eliminate the quadratic terms in the equation (4.1). Let

$$g := h + \mathcal{O}[h, b^{++}]h + \mathcal{O}[h, b^{+-}]\bar{h} + \mathcal{O}[\bar{h}, b^{--}]\bar{h} = h + \sum_{\iota_1 \iota_2 \in \Lambda} \mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}, \quad (4.3)$$

where  $\Lambda := \{++, +-, --\}$ ,  $h^+ := h$ ,  $h^- := \bar{h}$  and

$$b^{\iota_1 \iota_2}(\xi, \eta) := \frac{i q^{\iota_1 \iota_2}(\xi, \eta)}{\langle\xi+\eta\rangle - \iota_1\langle\xi\rangle - \iota_2\langle\eta\rangle}, \quad \iota_1 \iota_2 \in \Lambda. \quad (4.4)$$

We remark that for any  $\xi, \eta \in \mathbb{R}$ ,

$$|\langle\xi+\eta\rangle \pm \langle\xi\rangle \pm \langle\eta\rangle| \geq (\langle\xi+\eta\rangle + \langle\xi\rangle + \langle\eta\rangle)^{-1} > 0.$$

By (4.3) and (4.4), equation (4.1) is changed into

$$g_t + i\langle\partial_x\rangle g = \mathcal{N}(h), \quad (4.5)$$

where  $\mathcal{N}(h)$  denotes cubic nonlinear term,

$$\begin{aligned} \mathcal{N}(h) &:= \sum_{\iota_1 \iota_2 \in \Lambda} \mathcal{O}[\mathcal{O}[h^{\iota_1}, q^{\iota_1 \iota_2}]h^{\iota_2}, b^{++}]h + \sum_{\iota_1 \iota_2 \in \Lambda} \mathcal{O}[h, b^{++}]\mathcal{O}[h^{\iota_1}, q^{\iota_1 \iota_2}]h^{\iota_2} \\ &\quad + \sum_{\iota_1 \iota_2 \in \Lambda} \mathcal{O}[\mathcal{O}[h^{\iota_1}, q^{\iota_1 \iota_2}]h^{\iota_2}, b^{+-}]\bar{h} + \sum_{\iota_1 \iota_2 \in \Lambda} \mathcal{O}[h, b^{+-}]\overline{\mathcal{O}[h^{\iota_1}, q^{\iota_1 \iota_2}]h^{\iota_2}} \\ &\quad + \sum_{\iota_1 \iota_2 \in \Lambda} \mathcal{O}[\overline{\mathcal{O}[h^{\iota_1}, q^{\iota_1 \iota_2}]h^{\iota_2}}, b^{--}]\bar{h} + \sum_{\iota_1 \iota_2 \in \Lambda} \mathcal{O}[\bar{h}, b^{--}]\overline{\mathcal{O}[h^{\iota_1}, q^{\iota_1 \iota_2}]h^{\iota_2}}. \end{aligned} \quad (4.6)$$

Let  $w$  be the linear profile of  $g$ , that is

$$w(t) := e^{it\langle\partial_x\rangle} g(t), \quad (4.7)$$

then from (4.5),  $w$  satisfies

$$w_t = e^{it\langle\partial_x\rangle} (\partial_t + i\langle\partial_x\rangle)g = e^{it\langle\partial_x\rangle} \mathcal{N}(h). \quad (4.8)$$

Now we state the main result of this section.

**Proposition 4.1** Let  $h \in C([0, T]; H^N)$  be the solution of (4.1), and  $g, w$  be given by (4.3), (4.7), respectively. Assume that

$$\|h(0)\|_{H^N} + \|xh(0)\|_{H^{N_1+1}} + \|\langle \xi \rangle^{N_1+10} \widehat{h(0)}\|_{L^\infty} \leq \epsilon_0, \quad (4.9)$$

and

$$\begin{aligned} \sup_{t \in [0, T]} [(1+t)^{-p_0} \|h(t)\|_{H^N} + (1+t)^{-p_0} \|\Gamma h(t)\|_{H^{N_1}} + \|\langle \xi \rangle^{N_1+10} \widehat{w}(t)\|_{L^\infty} \\ + (1+t)^{1/2} \|h(t)\|_{W^{N_1+10, \infty}}] \leq \epsilon_1, \end{aligned} \quad (4.10)$$

where  $N = 300$ ,  $N_1 = 15$ ,  $0 < p_0 < 10^{-3}$  and  $0 < \epsilon_0 \ll \epsilon_1 \ll 1$ . Then we have

$$\sup_{t \in [0, T]} [(1+t)^{-p_0} \|xw(t)\|_{H^{N_1-4}}] \lesssim \epsilon_0 + \epsilon_1^2, \quad (4.11)$$

$$\sup_{t \in [0, T]} \|\langle \xi \rangle^{N_1+10} \widehat{w}(t)\|_{L^\infty} \lesssim \epsilon_0 + \epsilon_1^3, \quad (4.12)$$

$$\sup_{t \in [0, T]} [(1+t)^{1/2} \|h(t)\|_{W^{N_1+10, \infty}}] \lesssim \epsilon_0 + \epsilon_1^2. \quad (4.13)$$

To prove Proposition 4.1, we need to construct a new linear dispersive estimate for the solution of (4.1).

**Lemma 4.2** For all  $t \geq 0$ , there holds that

$$\|e^{\pm it\langle \partial_x \rangle} f\|_{L^\infty} \lesssim (1+t)^{-1/2} \|\widehat{f}\|_{L^\infty} + (1+t)^{-5/8} (\|f\|_{H^2} + \|xf\|_{H^1}). \quad (4.14)$$

The proof for this estimate is given in Lemma A.1 of the appendix. Let  $f = w$  (or  $\bar{w}$ ) in (4.14), Lemma 4.2 shows that the  $L^\infty$  norm of the solution  $g$  is controlled by the  $L^\infty$  norm of  $\widehat{w}$  and the Sobolev norms of  $w$  and  $xw$ . The estimates for these norms are presented in the following subsections.

#### 4.1 Proof of (4.11)

We need the following isotropic multiplier estimate for  $\mathcal{O}[h^{\ell_1}, q^{\ell_1 \ell_2}]h^{\ell_2}$  and  $\mathcal{N}(h)$ .

**Lemma 4.3** Let  $m(\xi, \eta)$  be a Fourier multiplier satisfying

$$\|m\|_{L^2(\mathbb{R}^2)} + \|\partial_\xi^2 m\|_{L^2(\mathbb{R}^2)} + \|\partial_\eta^2 m\|_{L^2(\mathbb{R}^2)} \lesssim 1, \quad (4.15)$$

then for any  $p_0, p_1, p_2 \in [1, +\infty]$  with  $p_0^{-1} = p_1^{-1} + p_2^{-1}$ , we have

$$\|\mathcal{O}[f_1, m]f_2\|_{L^{p_0}(\mathbb{R})} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}.$$

For the proof of this multiplier lemma, see Lemma B.2 in the appendix.

**Lemma 4.4** Under the same assumptions as Proposition 4.1, there hold

$$\|\mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}\|_{H^{N-5}} \lesssim \epsilon_1^2(1+t)^{p_0-1/2}, \quad (4.16)$$

$$\|\mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}\|_{W^{N_1+10,\infty}} \lesssim \epsilon_1(1+t)^{p_0/2-1/4}\|h\|_{W^{N_1+10,\infty}}, \quad (4.17)$$

$$\|\Gamma \mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}\|_{H^{N_1-5}} \lesssim \epsilon_1^2(1+t)^{p_0-1/2}, \quad (4.18)$$

where  $\iota_1 \iota_2 \in \Lambda = \{++, +-, --\}$ . Moreover, we have

$$\|x\mathcal{N}(h)\|_{H^{N_1-5}} \lesssim \epsilon_1^3(1+t)^{p_0}. \quad (4.19)$$

*Proof.* It follows from (4.10) that

$$\|h\|_{H^N} \lesssim \epsilon_1(1+t)^{p_0}, \quad \|\Gamma h\|_{H^{N_1}} \lesssim \epsilon_1(1+t)^{p_0}, \quad \|h\|_{W^{N_1+10,\infty}} \lesssim \epsilon_1(1+t)^{-1/2}. \quad (4.20)$$

By the definition (1.12),

$$\mathcal{F}(\langle \partial_x \rangle^{N-5} \mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2})(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} m^{\iota_1 \iota_2}(\xi - \eta, \eta) (\langle \xi - \eta \rangle^N + \langle \eta \rangle^N) \widehat{h^{\iota_1}}(\xi - \eta) \widehat{h^{\iota_2}}(\eta) d\eta$$

with

$$m^{\iota_1 \iota_2}(\xi - \eta, \eta) := \frac{\langle \xi \rangle^{N-5} b^{\iota_1 \iota_2}(\xi - \eta, \eta)}{\langle \xi - \eta \rangle^N + \langle \eta \rangle^N}.$$

Note that

$$\left| \partial_{\xi}^{a_1} \partial_{\eta}^{a_2} \left[ \frac{1}{\langle \xi + \eta \rangle \pm \langle \xi \rangle \pm \langle \eta \rangle} \right] \right| \lesssim \max(\langle \xi + \eta \rangle, \langle \xi \rangle, \langle \eta \rangle), \quad a_1, a_2 \geq 0, \quad (4.21)$$

then we deduce from (4.2) and (4.4) that

$$|\partial_{\xi}^{a_1} \partial_{\eta}^{a_2} b^{\iota_1 \iota_2}(\xi, \eta)| \lesssim (\max(\langle \xi + \eta \rangle, \langle \xi \rangle, \langle \eta \rangle))^3. \quad (4.22)$$

In view of (4.22), it is easy to check that  $m^{\iota_1 \iota_2}(\xi, \eta)$  satisfies (4.15), then Lemma 4.3 shows

$$\|\mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}\|_{H^{N-5}} \lesssim \|h^{\iota_1}\|_{H^N} \|h^{\iota_2}\|_{L^\infty} + \|h^{\iota_1}\|_{L^\infty} \|h^{\iota_2}\|_{H^N} \lesssim \epsilon_1^2(1+t)^{p_0-1/2}, \quad (4.23)$$

where we have used (4.20) in the last step. Hence, the bound (4.16) follows.

For (4.17), using the interpolation inequality,

$$\|h\|_{W^{N_1+15,\infty}} \lesssim \|h\|_{W^{N_1+10,\infty}}^{1/2} \|h\|_{W^{N_1+20,\infty}}^{1/2} \lesssim \epsilon_1(1+t)^{p_0/2-1/4},$$

then by Lemma 4.3, we have

$$\|\mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}\|_{W^{N_1+10,\infty}} \lesssim \|h\|_{L^\infty} \|h\|_{W^{N_1+15,\infty}} \lesssim \epsilon_1(1+t)^{p_0/2-1/4} \|h\|_{L^\infty}.$$

To prove (4.18), we first consider the case  $\iota_1 \iota_2 = ++$ . A direct computation gives

$$\begin{aligned} \mathcal{F}(\Gamma \mathcal{O}[h, b^{++}]h)(\xi) &= \mathcal{F}(\mathcal{O}[\Gamma h, b^{++}]h)(\xi) + \mathcal{F}(\mathcal{O}[h, b^{++}]\Gamma h)(\xi) \\ &\quad + \frac{i}{2\pi} \int_{\mathbb{R}} \partial_{\xi} b^{++}(\xi - \eta, \eta) \widehat{h}_t(\xi - \eta) \widehat{h}(\eta) d\eta \\ &\quad + \frac{i}{2\pi} \int_{\mathbb{R}} (\partial_{\xi} b^{++}(\xi - \eta, \eta) + \partial_{\eta} b^{++}(\xi - \eta, \eta)) \widehat{h}(\xi - \eta) \widehat{h}_t(\eta) d\eta. \end{aligned} \quad (4.24)$$

From the equation (4.1) and the bound (4.20), it is easy to see

$$\|h_t\|_{H^{N-1}} \lesssim \epsilon_1(1+t)^{p_0}, \quad \|h_t\|_{L^\infty} \lesssim \epsilon_1(1+t)^{-1/2}.$$

Then using (4.22)–(4.24) and Lemma 4.3, we obtain

$$\begin{aligned} \|\Gamma\mathcal{O}[h, b^{++}]h\|_{H^{N_1-5}} &\lesssim \|\Gamma h\|_{H^{N_1}} \|h\|_{W^{N_1, \infty}} + \|h_t\|_{H^{N_1}} \|h\|_{L^\infty} + \|h_t\|_{L^\infty} \|h\|_{H^{N_1}} \\ &\lesssim \epsilon_1^2(1+t)^{p_0-1/2}, \end{aligned}$$

which proves (4.18) for  $\iota_1\iota_2 = ++$ . The proof for  $\iota_1\iota_2 = +-,--$  is the same as above.

In order to prove (4.19), it suffices to show that each term in (4.6) satisfies the bound (4.19). Here, we only consider the term  $\mathcal{O}[\mathcal{O}[h, q^{++}]h, b^{++}]h$  in detail. Note that

$$\mathcal{F}(\mathcal{O}[\mathcal{O}[h, q^{++}]h, b^{++}]h)(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} b^{++}(\eta, \xi - \eta) \mathcal{F}(\mathcal{O}[h, q^{++}]h)(\eta) (\mathcal{F}h)(\xi - \eta) d\eta.$$

Applying  $\partial_\xi$  to this identity yields

$$x\mathcal{O}[\mathcal{O}[h, q^{++}]h, b^{++}]h = A_1 + A_2,$$

where

$$\begin{aligned} \widehat{A}_1(\xi) &:= \frac{i}{2\pi} \int_{\mathbb{R}} \partial_\xi b^{++}(\eta, \xi - \eta) \mathcal{F}(\mathcal{O}[h, q^{++}]h)(\eta) (\mathcal{F}h)(\xi - \eta) d\eta, \\ \widehat{A}_2(\xi) &:= \frac{i}{2\pi} \int_{\mathbb{R}} b^{++}(\eta, \xi - \eta) \mathcal{F}(\mathcal{O}[h, q^{++}]h)(\eta) \partial_\xi (\mathcal{F}h)(\xi - \eta) d\eta. \end{aligned}$$

Then using Lemma 4.3, (4.20), (4.22) and (4.2), we have

$$\|A_1\|_{H^{N_1-5}} \lesssim \|\mathcal{O}[h, q^{++}]h\|_{H^{N_1}} \|h\|_{W^{N_1, \infty}} \lesssim \|h\|_{W^{N_1+10, \infty}}^2 \|h\|_{H^N} \lesssim \epsilon_1^3(1+t)^{p_0-1}.$$

For the term  $A_2$ , Proposition 3.2 and (1.8) yield

$$\|xh\|_{H^{N_1}} \sim \|xU\|_{H^{N_1}} \lesssim \epsilon_1(1+t)^{1+p_0},$$

hence, we obtain

$$\|A_2\|_{H^{N_1-5}} \lesssim \|\mathcal{O}[h, q^{++}]h\|_{W^{N_1, \infty}} \|xh\|_{H^{N_1}} \lesssim \|h\|_{W^{N_1+10, \infty}}^2 \|xh\|_{H^{N_1}} \lesssim \epsilon_1^3(1+t)^{p_0}.$$

Therefore, we conclude that

$$\|x\mathcal{O}[\mathcal{O}[h, q^{++}]h, b^{++}]h\|_{H^{N_1-5}} \lesssim \epsilon_1^3(1+t)^{p_0},$$

and the desired bound (4.19) thus follows.  $\square$

*Proof of (4.11).* To estimate  $xw$ , an important tool is introducing the vector filed

$$\widetilde{\Gamma} := t\partial_x - i\langle \partial_x \rangle x, \tag{4.25}$$

which satisfies

$$\langle \xi \rangle \partial_\xi (e^{it\langle \xi \rangle} \mathcal{F}g) = e^{it\langle \xi \rangle} \mathcal{F}(\tilde{\Gamma}g). \quad (4.26)$$

Thus,  $\|xw\|_{H^{s+1}} \sim \|\tilde{\Gamma}g\|_{H^s}$ . Moreover, the relationship between  $\tilde{\Gamma}$  and the homogeneous vector field operator  $\Gamma = x\partial_t + t\partial_x$  is

$$\tilde{\Gamma}g = \Gamma g - x(\partial_t + i\langle \partial_x \rangle)g + \frac{i\partial_x}{\langle \partial_x \rangle}g = \Gamma g - x\mathcal{N}(h) + \frac{i\partial_x}{\langle \partial_x \rangle}g.$$

Using the bounds (2.2), (3.2), (4.16) and (4.18), we deduce

$$\begin{aligned} \|g\|_{H^{N-5}} &\lesssim \|h\|_{H^{N-5}} + \sum_{\iota_1 \iota_2 \in \Lambda} \|\mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}\|_{H^{N-5}} \lesssim (\epsilon_0 + \epsilon_1^2)(1+t)^{p_0}, \\ \|\Gamma g\|_{H^{N_1-5}} &\lesssim \|\Gamma h\|_{H^{N_1-5}} + \sum_{\iota_1 \iota_2 \in \Lambda} \|\Gamma \mathcal{O}[h^{\iota_1}, b^{\iota_1 \iota_2}]h^{\iota_2}\|_{H^{N_1-5}} \lesssim (\epsilon_0 + \epsilon_1^2)(1+t)^{p_0}. \end{aligned} \quad (4.27)$$

Hence, combining (4.19) and the above estimates, we obtain

$$\|\tilde{\Gamma}g\|_{H^{N_1-5}} \lesssim \|\Gamma g\|_{H^{N_1-5}} + \|x\mathcal{N}(h)\|_{H^{N_1-5}} + \|g\|_{H^{N_1-5}} \lesssim (\epsilon_0 + \epsilon_1^2)(1+t)^{p_0}.$$

Thanks to the identity (4.26), there holds

$$\|xw\|_{H^{N_1-4}} = \|\tilde{\Gamma}g\|_{H^{N_1-5}} \lesssim (\epsilon_0 + \epsilon_1^2)(1+t)^{p_0}.$$

The proof of (4.11) is completed.  $\square$

## 4.2 Proof of (4.12)

Now we consider the  $L^\infty$  bound for  $\hat{w}$  in low order norm and present the proof of (4.12). Indeed, we will be devoted in proving a more stronger result in this subsection.

**Proposition 4.5** *Under the same assumption as Proposition 4.1, there exists  $\delta > 0$  such that*

$$\sup_{0 \leq t_1 \leq t_2 \leq T} (1+t_1)^\delta \|\langle \xi \rangle^{N_1+10} e^{i\vartheta(t_1, \xi)} \hat{w}(t_1, \xi) - \langle \xi \rangle^{N_1+10} e^{i\vartheta(t_2, \xi)} \hat{w}(t_2, \xi)\|_{L^\infty} \lesssim \epsilon_1^3, \quad (4.28)$$

where  $w$  is defined by (4.7), and  $\vartheta$  is a real-valued function given by (4.37).

Once Theorem 1.1 is proved, the above proposition implies that the function

$$\langle \xi \rangle^{N_1+10} e^{i\vartheta(t, \xi)} \hat{w}(t, \xi)$$

forms a Cauchy family as  $t \rightarrow \infty$  in  $L^\infty$ , so there exists a unique  $w_\infty(\xi) \in L^\infty$  such that

$$\sup_{t \geq 0} [(1+t)^\delta \|\langle \xi \rangle^{N_1+10} e^{i\vartheta(t, \xi)} \hat{w}(t, \xi) - w_\infty(\xi)\|_{L^\infty}] \lesssim \epsilon_0.$$

This result says the solution of the equation (4.5) tends to a nonlinear asymptotic state as  $t \rightarrow \infty$ , thus such equation possesses a modified scattering behavior with corrected phase  $\vartheta(t, \xi)$ . Assuming Proposition 4.5 holds, we now show the proof of (4.12).

*Proof of (4.12).* By setting  $t_1 = 0$  and  $t_2 = t$  in the estimate (4.28), we see

$$\sup_{t \in [0, T]} \|\langle \xi \rangle^{N_1+10} \widehat{w}(t, \xi)\|_{L^\infty} \lesssim \epsilon_1^3 + \|\langle \xi \rangle^{N_1+10} \widehat{w(0)}\|_{L^\infty} = \epsilon_1^3 + \|\langle \xi \rangle^{N_1+10} \widehat{g(0)}\|_{L^\infty},$$

then the bound (4.12) follows immediately, provided that we can show

$$\|\langle \xi \rangle^{N_1+10} \widehat{g(0)}\|_{L^\infty} \lesssim \epsilon_0. \quad (4.29)$$

Indeed, note that from (4.3),

$$g(0) = h_0 + \mathcal{O}[h_0, b^{++}]h_0 + \mathcal{O}[h_0, b^{+-}]\overline{h_0} + \mathcal{O}[\overline{h_0}, b^{--}]\overline{h_0}, \quad h_0 := h(0).$$

Using the initial bound (4.9), we deduce that, for all  $\iota_1 \iota_2 \in \Lambda$ ,

$$\|\langle \xi \rangle^{N_1+10} \mathcal{F}(\mathcal{O}[h_0^{\iota_1}, b^{\iota_1 \iota_2}]h_0^{\iota_2})(\xi)\|_{L^\infty} \lesssim \|\mathcal{O}[h_0^{\iota_1}, b^{\iota_1 \iota_2}]h_0^{\iota_2}\|_{W^{N_1+10,1}} \lesssim \|h_0\|_{H^N}^2 \lesssim \epsilon_0^2.$$

Therefore, the bound (4.29) follows from the above estimate and (4.9).  $\square$

From now on, we concentrate on the proof of Proposition 4.5. Rewrite the nonlinear term of the equation (4.5) as

$$\mathcal{N}(h) = \mathcal{N}(g) + \mathcal{N}_R, \quad \mathcal{N}_R := \mathcal{N}(h) - \mathcal{N}(g), \quad (4.30)$$

then the profile  $w$  satisfies

$$w_t = e^{it\langle \partial_x \rangle} \mathcal{N}(g) + e^{it\langle \partial_x \rangle} \mathcal{N}_R, \quad (4.31)$$

where  $\mathcal{N}(g)$  denotes cubic term and  $\mathcal{N}_R$  is quartic term. From the definition (4.6), the first nonlinear term in the RHS of (4.31) can be expanded as

$$e^{it\langle \xi \rangle} \widehat{\mathcal{N}(g)}(\xi) := i(2\pi)^{-2} [I^{++-}(t, \xi) + I^{+-}(t, \xi) + I^{++}(t, \xi) + I^{--}(t, \xi)],$$

where

$$I^{\iota_1 \iota_2 \iota_3}(t, \xi) := \int_{\mathbb{R}^2} c^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma) e^{it\Psi^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma)} \widehat{w^{\iota_1}}(t, \xi - \eta) \widehat{w^{\iota_2}}(t, \eta - \sigma) \widehat{w^{\iota_3}}(t, \sigma) d\eta d\sigma \quad (4.32)$$

with  $\iota_1 \iota_2 \iota_3 \in \mathcal{T} := \{+ + -, + --, + ++, ---\}$  and  $w^+ := w$ ,  $w^- := \overline{w}$ . If there is no confusion occurs, we also simply write (4.32) as

$$I^{\iota_1 \iota_2 \iota_3} = \int_{\mathbb{R}^2} c^{\iota_1 \iota_2 \iota_3} e^{it\Psi^{\iota_1 \iota_2 \iota_3}} \widehat{w^{\iota_1}}(\xi - \eta) \widehat{w^{\iota_2}}(\eta - \sigma) \widehat{w^{\iota_3}}(\sigma) d\eta d\sigma.$$

The phase  $\Psi^{\iota_1\iota_2\iota_3}$  is defined by

$$\Psi^{\iota_1\iota_2\iota_3}(\xi, \eta, \sigma) := \langle \xi \rangle - \iota_1 \langle \xi - \eta \rangle - \iota_2 \langle \eta - \sigma \rangle - \iota_3 \langle \sigma \rangle, \quad (4.33)$$

and the symbols  $c^{\iota_1\iota_2\iota_3}$  are

$$\begin{aligned} ic^{++-}(\xi, \eta, \sigma) &:= b^{++}(\eta, \xi - \eta)q^{+-}(\eta - \sigma, \sigma) + b^{++}(\xi - \eta, \eta)q^{+-}(\eta - \sigma, \sigma) \\ &\quad + b^{+-}(\xi - \sigma, \sigma)q^{++}(\xi - \eta, \eta - \sigma) + b^{+-}(\xi - \eta, \eta)q^{+-}(-\sigma, \sigma - \eta) \\ &\quad + b^{--}(\xi - \sigma, \sigma)q^{--}(\eta - \xi, \sigma - \eta) + b^{--}(\sigma, \xi - \sigma)q^{--}(\eta - \xi, \sigma - \eta), \\ ic^{+--}(\xi, \eta, \sigma) &:= b^{++}(\eta, \xi - \eta)q^{--}(\eta - \sigma, \sigma) + b^{++}(\xi - \eta, \eta)q^{--}(\eta - \sigma, \sigma) \\ &\quad + b^{+-}(\xi - \sigma, \sigma)q^{+-}(\xi - \eta, \eta - \sigma) + b^{+-}(\xi - \eta, \eta)q^{++}(\sigma - \eta, -\sigma) \\ &\quad + b^{--}(\xi - \sigma, \sigma)q^{+-}(\sigma - \eta, \eta - \xi) + b^{--}(\sigma, \xi - \sigma)q^{+-}(\sigma - \eta, \eta - \xi), \\ ic^{+++}(\xi, \eta, \sigma) &:= b^{++}(\eta, \xi - \eta)q^{++}(\eta - \sigma, \sigma) + b^{++}(\xi - \eta, \eta)q^{++}(\eta - \sigma, \sigma) \\ &\quad + b^{+-}(\xi - \eta, \eta)q^{--}(\sigma - \eta, -\sigma), \\ ic^{---}(\xi, \eta, \sigma) &:= b^{+-}(\xi - \sigma, \sigma)q^{--}(\xi - \eta, \eta - \sigma) + b^{--}(\xi - \sigma, \sigma)q^{++}(\eta - \xi, \sigma - \eta) \\ &\quad + b^{--}(\sigma, \xi - \sigma)q^{++}(\eta - \xi, \sigma - \eta), \end{aligned}$$

where  $q^{\iota_1\iota_2}$  and  $b^{\iota_1\iota_2}$  are given by (4.2) and (4.4). Therefore, we conclude that

$$\widehat{w}_t(t, \xi) = \sum_{\iota_1\iota_2\iota_3 \in \mathcal{T}} i(2\pi)^{-2} I^{\iota_1\iota_2\iota_3}(t, \xi) + e^{it\langle \xi \rangle} \widehat{\mathcal{N}_R}(t, \xi). \quad (4.34)$$

For the phase  $\Psi^{\iota_1\iota_2\iota_3}$ , we can compute the space-time resonance set ([6])

$$\{(\xi, \eta, \sigma); \Psi^{\iota_1\iota_2\iota_3}(\xi, \eta, \sigma) = \Psi_\eta^{\iota_1\iota_2\iota_3}(\xi, \eta, \sigma) = \Psi_\sigma^{\iota_1\iota_2\iota_3}(\xi, \eta, \sigma) = 0\}.$$

Indeed, it is easy to check that the only space-time resonance is in the case  $\iota_1\iota_2\iota_3 = ++-$ , and the resonant set is  $(\xi, \eta, \sigma) = (\xi, 0, -\xi)$ . A direct computation gives

$$\begin{aligned} c^*(\xi) &:= c^{++-}(\xi, 0, -\xi) \\ &= \xi^2 \left[ 2\langle \xi \rangle - \frac{(2\langle \xi \rangle + \langle 2\xi \rangle) \cdot (\langle \xi \rangle \langle 2\xi \rangle + \xi^2 + \langle \xi \rangle^2)^2}{6\langle \xi \rangle \langle 2\xi \rangle} + \frac{(\langle \xi \rangle \langle 2\xi \rangle - \langle \xi \rangle^2 - \xi^2)^2}{2(2\langle \xi \rangle + \langle 2\xi \rangle) \langle \xi \rangle \langle 2\xi \rangle} \right], \end{aligned} \quad (4.35)$$

thus,

$$c^*(0) = c_\xi^*(0) = 0, \quad |c^*(\xi)| \lesssim \xi^2 \langle \xi \rangle^3, \quad |c_\xi^*(\xi)| \lesssim |\xi| \langle \xi \rangle^3. \quad (4.36)$$

Define

$$\vartheta(t, \xi) := -\frac{c^*(\xi) \langle \xi \rangle^3}{2\pi} \int_0^t \frac{|\widehat{w}(s, \xi)|^2}{s+1} ds, \quad (4.37)$$

then it follows from (4.34) and (4.37) that

$$\begin{aligned}
\partial_t[e^{i\vartheta(t,\xi)}\widehat{w}(t,\xi)] &= e^{i\vartheta(t,\xi)}i\vartheta_t(t,\xi)\widehat{w}(t,\xi) + e^{i\vartheta(t,\xi)}\partial_t\widehat{w}(t,\xi) \\
&= \frac{i}{4\pi^2}e^{i\vartheta(t,\xi)}\left[I^{++-}(t,\xi) - 2\pi\frac{c^*(\xi)\langle\xi\rangle^3|\widehat{w}(t,\xi)|^2\widehat{w}(t,\xi)}{1+t}\right] \\
&\quad + \frac{i}{4\pi^2}e^{i\vartheta(t,\xi)}[I^{---}(t,\xi) + I^{+++}(t,\xi) + I^{--+}(t,\xi)] \\
&\quad + e^{i\vartheta(t,\xi)}e^{it\langle\xi\rangle}\widehat{\mathcal{N}_R}(t,\xi).
\end{aligned} \tag{4.38}$$

Now we make frequency decomposition. Let

$$\begin{aligned}
I_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(s, \xi) &:= \int_{\mathbb{R}^2} c_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma) e^{is\Psi^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma)} \\
&\quad \cdot \widehat{P_{k_1} w^{\iota_1}}(s, \xi - \eta) \widehat{P_{k_2} w^{\iota_2}}(s, \eta - \sigma) \widehat{P_{k_3} w^{\iota_3}}(s, \sigma) d\eta d\sigma,
\end{aligned} \tag{4.39}$$

where

$$c_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma) := c^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma) \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta - \sigma) \varphi_{k_3}(\sigma).$$

For our proof, it is sufficient to use the following bound for this symbol

$$|\partial_\xi^{a_1} \partial_\eta^{a_2} \partial_\sigma^{a_3} c_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma)| \lesssim 2^{5 \max(k_1, k_2, k_3)_+}, \quad a_1, a_2, a_3 \geq 0, \tag{4.40}$$

where  $a_+ := \max\{a, 0\}$ . (4.40) can be obtained from the definitions of  $c^{\iota_1 \iota_2 \iota_3}$  and a direct computation. The detailed expressions of  $c^{\iota_1 \iota_2 \iota_3}$  won't play an important role in our succeeding arguments.

In virtue of (4.38)–(4.39), in order to prove (4.28), it suffices to prove there exists  $\delta > 0$  such that

$$\begin{aligned}
\sum_{k_1, k_2, k_3 \in \mathbb{Z}} \left| \int_{t_1}^{t_2} e^{i\vartheta(s, \xi)} \left[ I_{k_1 k_2 k_3}^{++-}(s, \xi) - 2\pi \frac{c^*(\xi)\langle\xi\rangle^3 \widehat{P_{k_1} w}(s, \xi) \widehat{P_{k_2} w}(s, \xi) \widehat{P_{k_3} \bar{w}}(s, -\xi)}{s+1} \right] ds \right| \\
\lesssim \epsilon_1^3 2^{-\delta m} 2^{-(N_1+10)k_+},
\end{aligned} \tag{4.41}$$

and for  $\iota_1 \iota_2 \iota_3 \in \{+--, +++, ---\}$ ,

$$\sum_{k_1, k_2, k_3 \in \mathbb{Z}} \left| \int_{t_1}^{t_2} e^{i\vartheta(s, \xi)} I_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(s, \xi) ds \right| \lesssim \epsilon_1^3 2^{-\delta m} 2^{-(N_1+10)k_+}, \tag{4.42}$$

where  $|\xi| \sim 2^k$ ,  $k \in \mathbb{Z}$ , and  $t_1, t_2 \in [2^m - 2, 2^{m+1}] \cap [0, T]$ ,  $m = 1, 2, 3, \dots$ . Moreover, we shall also prove

$$\left| \langle \xi \rangle^{N_1+10} \int_{t_1}^{t_2} e^{i\vartheta(s, \xi)} e^{is\langle\xi\rangle} \widehat{\mathcal{N}_R}(s, \xi) ds \right| \lesssim \epsilon_1^4 (1 + t_1)^{-\delta}.$$

To prove these bounds, we need some basic estimates for the localized function  $P_k w$ , which are given in the following lemma.

**Lemma 4.6** *With the same assumption as Proposition 4.1, we have*

$$\|\widehat{P_k w}\|_{L^\infty} \lesssim \epsilon_1 2^{-(N_1+10)k_+}, \quad (4.43)$$

$$\|\partial_\xi \widehat{P_k w}\|_{L^2} \lesssim \epsilon_1 2^{p_0 m} 2^{-(N_1-4)k_+}, \quad (4.44)$$

$$\|P_k w\|_{L^2} \lesssim \epsilon_1 2^{p_0 m} 2^{-(N-5)k_+}, \quad (4.45)$$

$$\|e^{\pm is\langle \partial_x \rangle} P_k w\|_{L^\infty} \lesssim \epsilon_1 2^{-m/2}, \quad (4.46)$$

$$\|e^{\pm is\langle \partial_x \rangle} P_k w\|_{L^\infty} \lesssim \epsilon_1 2^k 2^{-(N_1+10)k_+}, \quad (4.47)$$

$$\|e^{\pm is\langle \partial_x \rangle} P_k w\|_{L^2} \lesssim \epsilon_1 2^{k/2} 2^{-(N_1+10)k_+}, \quad (4.48)$$

$$\|\partial_s \widehat{P_k w}\|_{L^2} \lesssim \epsilon_1^3 2^{p_0 m} 2^{-m} 2^{-(N-7)k_+}, \quad (4.49)$$

where  $s \in [2^m - 2, 2^{m+1}]$ ,  $m \in \mathbb{N}$  and  $k_+ = \max\{k, 0\}$ .

*Proof.* The bounds (4.43), (4.44) follow from (4.10), (4.11), respectively. Using (4.3), (4.16) and (4.10), we can obtain

$$\|w\|_{H^{N-5}} \lesssim \epsilon_1 (1+t)^{p_0}, \quad (4.50)$$

which gives (4.45). The bound (4.46) follows from (4.14), (4.10), (4.11) and (4.50). Note that

$$|e^{\pm is\langle \partial_x \rangle} P_k w| = \frac{1}{2\pi} \left| \int_{\mathbb{R}} e^{ix\xi} e^{\pm is\langle \xi \rangle} \widehat{P_k w}(s, \xi) d\xi \right| \lesssim \|\widehat{P_k w}\|_{L^\infty} 2^k,$$

then (4.47) follows from (4.43). The estimate (4.48) is proved by the Plancherel's identity, Cauchy-Schwarz inequality and (4.43). For (4.49), we can obtain from (4.6) and (4.8) that

$$\|\partial_s w\|_{H^{N-7}} = \|\mathcal{N}(h)\|_{H^{N-7}} \lesssim \|h\|_{H^N} \|h\|_{W^{N_1+10,\infty}}^2 \lesssim \epsilon_1^3 (1+s)^{-1+p_0},$$

so the desired bound (4.49) follows easily.  $\square$

We first show the bounds (4.41) and (4.42) in two simpler cases.

**Lemma 4.7** *The bounds (4.41) and (4.42) hold if we take the sum over those  $(k_1, k_2, k_3)$  satisfying*

$$\min(k_1, k_2, k_3) \leq -4m \quad \text{or} \quad \max(k_1, k_2, k_3) \geq m/200 - 100. \quad (4.51)$$

*Proof.* Using (4.40), (4.45) and Cauchy-Schwarz inequality, we see that

$$|I_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(s, \xi)| \lesssim \epsilon_1^3 2^{3p_0 m} 2^{5 \max(k_1, k_2, k_3)_+} \cdot 2^{\min(k_1, k_2, k_3)/2} \cdot 2^{-(N-5) \min(k_1, k_2, k_3)_+} \cdot 2^{-(N-5) \text{med}(k_1, k_2, k_3)_+} \cdot 2^{-(N-5) \max(k_1, k_2, k_3)_+}$$

for any  $\iota_1 \iota_2 \iota_3 \in \mathcal{T}$ . Using (4.36) and the  $L^\infty$  bound (4.43), there holds

$$\left| \frac{c^*(\xi) \langle \xi \rangle^3 \widehat{P_{k_1} w}(s, \xi) \widehat{P_{k_2} w}(s, \xi) \widehat{P_{k_3} w}(s, -\xi)}{s+1} \right|$$

$$\lesssim 2^{-m} \epsilon_1^3 2^{2k} 2^{6k_+} 2^{-3(N_1+10)k_+} \mathbf{1}_{[0,20]} \max(|k_1 - k|, |k_2 - k|, |k_3 - k|).$$

In virtue of (4.51), Lemma 4.7 thus follows.  $\square$

If one takes the sum for  $(k_1, k_2, k_3)$  which satisfies  $k_1, k_2, k_3 \in [-4m, m/200 - 100] \cap \mathbb{Z}$ , then there are at most  $Cm^3$  terms, which are summable as the desired estimate (4.41) or (4.42) has an exponential factor  $2^{-\delta m}$ . So in the following it is sufficient for us to prove that for fixed  $k_1, k_2, k_3$ , there exists  $\delta > 0$  such that

$$\begin{aligned} & \left| \int_{t_1}^{t_2} e^{i\vartheta(s, \xi)} [I_{k_1 k_2 k_3}^{++-}(s, \xi) - 2\pi \frac{c^*(\xi)\langle \xi \rangle^3 \widehat{P_{k_1} w}(s, \xi) \widehat{P_{k_2} w}(s, \xi) \widehat{P_{k_3} \bar{w}}(s, -\xi)}{s+1}] ds \right| \\ & \lesssim \epsilon_1^3 2^{-\delta m} 2^{-(N_1+10)k_+}, \end{aligned} \quad (4.52)$$

and for  $\iota_1 \iota_2 \iota_3 \in \{+--\}, +++, ---\}$ ,

$$\left| \int_{t_1}^{t_2} e^{i\vartheta(s, \xi)} I_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(s, \xi) ds \right| \lesssim \epsilon_1^3 2^{-\delta m} 2^{-(N_1+10)k_+}. \quad (4.53)$$

**Lemma 4.8** *The estimate (4.53) holds if  $k_1, k_2, k_3 \in [-4m, m/200 - 100] \cap \mathbb{Z}$  and*

$$\min(k_1, k_2, k_3) + \text{med}(k_1, k_2, k_3) \leq -6m/5. \quad (4.54)$$

*If, in addition,*

$$\max(|k_1 - k|, |k_2 - k|, |k_3 - k|) \geq 21, \quad (4.55)$$

*then the estimate (4.52) also holds.*

*Proof.* Under the condition (4.54), we use (4.40), the  $L^\infty$  bound (4.43) to get

$$\begin{aligned} |I_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(s, \xi)| & \lesssim \epsilon_1^3 2^{5 \max(k_1, k_2, k_3)_+} \cdot 2^{\min(k_1, k_2, k_3)} 2^{\text{med}(k_1, k_2, k_3)} \\ & \quad \cdot 2^{-(N_1+10)k_1} \cdot 2^{-(N_1+10)k_2} \cdot 2^{-(N_1+10)k_3} \\ & \lesssim \epsilon_1^3 2^{(N_1+15) \max(k_1, k_2, k_3)_+} \cdot 2^{-6m/5} 2^{-(N_1+10)k_+} \\ & \lesssim \epsilon_1^3 2^{-31m/30} 2^{-(N_1+10)k_+} \end{aligned}$$

for any  $\iota_1 \iota_2 \iota_3 \in \mathcal{T}$ , where in the last step, we have also used

$$(N_1 + 15) \max(k_1, k_2, k_3) \leq (N_1 + 15)m/200 < m/6.$$

Therefore, the estimate (4.53) clearly holds. If, in addition, (4.55) holds, then

$$\varphi_k(\xi) \widehat{P_{k_1} w}(s, \xi) \widehat{P_{k_2} w}(s, \xi) \widehat{P_{k_3} \bar{w}}(s, -\xi) = 0,$$

so the estimate (4.52) follows.  $\square$

In view of the above two lemmas, in order to prove (4.41) and (4.42), it suffices to show the following proposition.

**Proposition 4.9** Let  $k \in \mathbb{Z}$ ,  $|\xi| \sim 2^k$  and  $t_1, t_2 \in [2^m - 2, 2^{m+1}] \cap [0, T]$ ,  $m \geq 25$  be an integer. Assume that  $k_1, k_2, k_3$  satisfies

$$k_1, k_2, k_3 \in [-4m, m/200 - 100] \cap \mathbb{Z}, \quad (4.56)$$

and

$$\min(k_1, k_2, k_3) + \text{med}(k_1, k_2, k_3) \geq -6m/5. \quad (4.57)$$

Then the estimates (4.52) and (4.53) are valid.

As mentioned before, in order to finish the proof of Proposition 4.5, we shall also prove

**Proposition 4.10** For any  $0 \leq t_1 \leq t_2 \leq T$ , there exists  $\delta > 0$  such that

$$\left| \langle \xi \rangle^{N_1+10} \int_{t_1}^{t_2} e^{i\vartheta(s, \xi)} e^{is\langle \xi \rangle} \widehat{\mathcal{N}_R}(s, \xi) ds \right| \lesssim \epsilon_1^4 (1 + t_1)^{-\delta}. \quad (4.58)$$

According to the above reductions, we see Proposition 4.5 follows easily from Propositions 4.9–4.10. Hence, the remaining part of this subsection is devoted to the proofs of these two propositions. The bound (4.52) is proven through Lemmas 4.14–4.16 below, depending on different cases between the sizes of the input and output frequencies, and the bound (4.53) is obtained by Lemma 4.17. In addition, we will establish the bound (4.58) with the help of Lemma 4.18. In the proofs, we will frequently use the following multiplier lemma.

**Lemma 4.11** There holds

$$\left| \int_{\mathbb{R}^2} m(\eta, \sigma) \widehat{f_1}(\eta) \widehat{f_2}(\sigma) \widehat{f_3}(-\eta - \sigma) d\eta d\sigma \right| \lesssim \|\mathcal{F}^{-1}m\|_{L^1} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}$$

with  $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$  and  $p_1, p_2, p_3 \in [1, +\infty]$ .

The proof of Lemma 4.11 can be found in [18]. To bound the  $L^1$  norm of  $\mathcal{F}^{-1}m$ , we usually use Lemma 4.12 below.

**Lemma 4.12** If  $m(\eta, \sigma)$  is a Fourier multiplier with  $\eta$  and  $\sigma$  localized in the size  $2^k$  and  $2^l$ , respectively, and satisfies

$$|\partial_\eta^a \partial_\sigma^b m| \lesssim A 2^{-ak} 2^{-bl} \quad (\text{resp. } A) \quad (4.59)$$

for any  $a, b = 0, 1, 2$ , then we have

$$\|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^2)} \lesssim A \quad (\text{resp. } A 2^k 2^l). \quad (4.60)$$

**Lemma 4.13** For any  $\lambda, \mu > 0$  and  $n \in \mathbb{N}$ , there holds that

$$\int_{\mathbb{R}^2} e^{i\lambda xy} \varphi(\mu^{-1}x) \varphi(\mu^{-1}y) dx dy = 2\pi\lambda^{-1} + \lambda^{-1-n}\mu^{-2n}O(1),$$

where  $\varphi$  is the smooth radial function used in the Littlewood-Paley decomposition. The implicit constant coming from the term  $O(1)$  depends only on  $n$  and  $\varphi$ .

Lemmas 4.12–4.13 are proved in the appendix (see Lemma B.3 and Lemma B.4).

**Lemma 4.14** The estimate (4.52) holds provided that

$$\max(|k_1 - k|, |k_2 - k|, |k_3 - k|) \leq 20. \quad (4.61)$$

*Proof.* It suffices to prove, for any  $s \in [t_1, t_2]$ , that

$$\left| I_{k_1 k_2 k_3}^{++-}(s, \xi) - 2\pi \frac{c^*(\xi)\langle\xi\rangle^3 \widehat{P_{k_1} w}(s, \xi) \widehat{P_{k_2} w}(s, \xi) \widehat{P_{k_3} \bar{w}}(s, -\xi)}{s+1} \right| \lesssim \epsilon_1^3 2^{-(1+\delta_1)m} 2^{-(N_1+10)k_+} \quad (4.62)$$

for some  $\delta_1 > 0$ . We split the proof into several steps.

**Step 1:**  $|\xi| \lesssim 2^{-m}$ . In this case, we use (4.40) and the  $L^\infty$  bound (4.43) to obtain

$$\begin{aligned} \text{LHS of (4.62)} &\lesssim \epsilon_1^3 2^{2k} 2^{5k_+} 2^{-3(N_1+10)k_+} + \epsilon_1^3 2^{-m} 2^{2k} 2^{6k_+} 2^{-3(N_1+10)k_+} \\ &\lesssim \epsilon_1^3 2^{-2m} 2^{-(3N_1+24)k_+}, \end{aligned}$$

which is better than the desired bound.

**Step 2:**  $|\xi| \gtrsim 2^{-m}$ . For the sake of convenience, we rewrite (by the change of variables  $\eta \rightarrow -\eta, \sigma \rightarrow -\xi - \sigma - \eta$ )

$$I_{k_1 k_2 k_3}^{++-} = \int_{\mathbb{R}^2} \tilde{c}_{k_1 k_2 k_3}^{++-}(\xi, \eta, \sigma) e^{is\Psi(\xi, \eta, \sigma)} \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \widehat{P_{k_3} \bar{w}}(-\xi - \eta - \sigma) d\eta d\sigma,$$

where

$$\tilde{c}^{++-}(\xi, \eta, \sigma) := c(\xi, -\eta, -\xi - \sigma - \eta), \quad (4.63)$$

$$\Psi(\xi, \eta, \sigma) := \langle\xi\rangle - \langle\xi + \eta\rangle - \langle\xi + \sigma\rangle + \langle\xi + \eta + \sigma\rangle. \quad (4.64)$$

Note that the set of space-time resonance for  $\Psi$  now reduces to  $(\xi, \eta, \sigma) = (\xi, 0, 0)$ . We see from (4.35) and (4.63) that

$$\tilde{c}^{++-}(\xi, 0, 0) = c(\xi, 0, -\xi) = c^*(\xi).$$

Let  $\bar{l}$  be the smallest integer satisfying  $2^{\bar{l}} \geq 2^{-9m/20}$ . Note that  $2^k \gtrsim 2^{-m}$  implies  $\bar{l} \leq k+10$ . Now, we decompose

$$I_{k_1 k_2 k_3}^{++-}(s, \xi) = \sum_{l_1, l_2 = \bar{l}}^{k+100} J_{l_1 l_2}(s, \xi), \quad (4.65)$$

where

$$J_{l_1 l_2}(s, \xi) := \int_{\mathbb{R}^2} \tilde{c}_{k_1 k_2 k_3}^{++-} e^{is\Psi} \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \widehat{P_{k_3} \bar{w}}(-\xi - \eta - \sigma) \varphi_{l_1}^{(\bar{l})}(\eta) \varphi_{l_2}^{(\bar{l})}(\sigma) d\eta d\sigma,$$

and

$$\varphi_l^{(l_0)}(\xi) := \begin{cases} \varphi(|\xi|/2^l) - \varphi(|\xi|/2^{l-1}), & l \geq l_0 + 1, \\ \varphi(|\xi|/2^{l_0}), & l = l_0. \end{cases} \quad (4.66)$$

In the following, we consider three different cases.

**Case 2a:**  $\sigma$  is away from the space-time resonance set. We aim to show that

$$|J_{l_1 l_2}(s, \xi)| \lesssim \epsilon_1^3 2^{-m} 2^{-\delta_1 m} 2^{-(N_1+10)k_+}, \quad l_2 \geq \max(l_1, \bar{l} + 1). \quad (4.67)$$

From (4.64), it is easy to see

$$|\partial_\eta \Psi| = \left| -\frac{\xi + \eta}{\langle \xi + \eta \rangle} + \frac{\xi + \eta + \sigma}{\langle \xi + \eta + \sigma \rangle} \right| \gtrsim 2^{l_2} 2^{-3k_+}, \quad (4.68)$$

whenever  $|\xi + \eta| \sim |\xi + \sigma| \sim |\xi + \eta + \sigma| \sim 2^k$  and  $|\sigma| \sim 2^{l_2}$ . With integration by parts in  $\eta$ , we have

$$|J_{l_1 l_2}(s, \xi)| \leq |J_{l_1 l_2,1}(s, \xi)| + |J_{l_1 l_2,2}(s, \xi)| + |J_{l_1 l_2,3}(s, \xi)|,$$

where

$$\begin{aligned} J_{l_1 l_2,1} &= \int_{\mathbb{R}^2} m_1 e^{is\Psi} \partial_\eta \widehat{P_{k_1} w}(s, \xi + \eta) \widehat{P_{k_2} w}(s, \xi + \sigma) \widehat{P_{k_3} \bar{w}}(s, -\xi - \eta - \sigma) d\eta d\sigma, \\ J_{l_1 l_2,2} &= \int_{\mathbb{R}^2} m_1 e^{is\Psi} \widehat{P_{k_1} w}(s, \xi + \eta) \widehat{P_{k_2} w}(s, \xi + \sigma) \partial_\eta \widehat{P_{k_3} \bar{w}}(s, -\xi - \eta - \sigma) d\eta d\sigma, \\ J_{l_1 l_2,3} &= \int_{\mathbb{R}^2} \partial_\eta m_1 e^{is\Psi} \widehat{P_{k_1} w}(s, \xi + \eta) \widehat{P_{k_2} w}(s, \xi + \sigma) \widehat{P_{k_3} \bar{w}}(s, -\xi - \eta - \sigma) d\eta d\sigma, \end{aligned}$$

with

$$m_1(\eta, \sigma) := \varphi_{l_1}^{(\bar{l})}(\eta) \varphi_{l_2}^{(\bar{l})}(\sigma) \cdot (s \partial_\eta \Psi)^{-1} \cdot \tilde{c}_{k_1 k_2 k_3}^{++-}.$$

Using (4.68) and the fact  $l_2 \geq l_1$ , we compute

$$|\partial_\eta^a \partial_\sigma^b [\varphi_{l_1}^{(\bar{l})}(\eta) \varphi_{l_2}^{(\bar{l})}(\sigma) (s \partial_\eta \Psi)^{-1}]| \lesssim 2^{-m} 2^{-l_2} 2^{3k_+} 2^{-al_1} 2^{-bl_2}, \quad a, b = 0, 1, 2.$$

Then by (4.59)–(4.60), we have

$$\|\mathcal{F}^{-1}[\varphi_{l_1}^{(\bar{l})}(\eta) \varphi_{l_2}^{(\bar{l})}(\sigma) (s \partial_\eta \Psi)^{-1}]\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-m} 2^{-l_2} 2^{3k_+}.$$

Recalling the bound (4.40) for  $\tilde{c}_{k_1 k_2 k_3}^{++-}$ , we deduce from Lemma 4.12 that

$$\|\mathcal{F}^{-1} \tilde{c}_{k_1 k_2 k_3}^{++-}\|_{L^1(\mathbb{R}^2)} \lesssim 2^{5k_+} 2^{2k}.$$

Combining the above two bounds give

$$\|\mathcal{F}^{-1} m_1\|_{L^1(\mathbb{R}^2)} \lesssim \|\mathcal{F}^{-1}[\varphi_{l_1}^{(\bar{l})}(\eta) \varphi_{l_2}^{(\bar{l})}(\sigma) (s \partial_\eta \Psi)^{-1}]\|_{L^1(\mathbb{R}^2)} \|\mathcal{F}^{-1} \tilde{c}_{k_1 k_2 k_3}^{++-}\|_{L^1(\mathbb{R}^2)}$$

$$\lesssim 2^{-m} 2^{-l_2} 2^{8k_+} 2^{2k}. \quad (4.69)$$

Similarly, we can obtain

$$\|\mathcal{F}^{-1}(\partial_\eta m_1)\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-m} 2^{-l_1} 2^{-l_2} 2^{8k_+} 2^{2k}.$$

Now, we apply Lemma 4.11 with

$$\begin{aligned}\widehat{\alpha}(\eta) &:= e^{-is\langle\xi+\eta\rangle} \partial_\eta \widehat{P_{k_1}w}(s, \xi + \eta),, \\ \widehat{\beta}(\sigma) &:= e^{-is\langle\xi+\sigma\rangle} \widehat{P_{k_2}w}(s, \xi + \sigma), \quad |\sigma| \sim 2^{l_2}, \\ \widehat{\gamma}(\zeta) &:= e^{is\langle-\xi+\zeta\rangle} \widehat{P_{k_3}w}(s, -\xi + \zeta),\end{aligned}$$

then

$$|J_{l_1 l_2, 1}| \lesssim \|\mathcal{F}^{-1}m_1\|_{L^1} \|\alpha\|_{L^2} \|\beta\|_{L^2} \|\gamma\|_{L^\infty}.$$

Using the fact  $|\sigma| \sim 2^{l_2}$ , (4.44), (4.43) and (4.46), we see

$$\|\alpha\|_{L^2} \lesssim \epsilon_1 2^{p_0 m} 2^{-(N_1-4)k_+}, \quad \|\beta\|_{L^2} \lesssim \epsilon_1 2^{l_2/2} 2^{-(N_1+10)k_+}, \quad \|\gamma\|_{L^\infty} \lesssim \epsilon_1 2^{-m/2}.$$

Therefore, these estimates and (4.69) lead to

$$\begin{aligned}|J_{l_1 l_2, 1}| &\lesssim 2^{-m} 2^{-l_2} 2^{8k_+} 2^{2k} \cdot \epsilon_1 2^{p_0 m} 2^{-(N_1-4)k_+} \cdot \epsilon_1 2^{l_2/2} 2^{-(N_1+10)k_+} \cdot \epsilon_1 2^{-m/2} \\ &= \epsilon_1^3 2^{-3m/2} 2^{p_0 m} 2^{-l_2/2} 2^{2k} 2^{-(2N_1-2)k_+}.\end{aligned}$$

Since  $2^{-l_2/2} \lesssim 2^{9m/40}$ ,  $J_{l_1 l_2, 1}$  can be bounded by  $\epsilon_1^3 2^{-51m/40} 2^{p_0 m} 2^{-(N_1+10)k_+}$ . With similar argument as above, we obtain the same bound for  $|J_{l_1 l_2, 2}|$ . For the term  $J_{l_1 l_2, 3}$ , we apply Lemma 4.11 with

$$\begin{aligned}\widehat{\tilde{\alpha}}(\eta) &:= e^{-is\langle\xi+\eta\rangle} \widehat{P_{k_1}w}(s, \xi + \eta), \quad |\eta| \lesssim 2^{l_1}, \\ \widehat{\tilde{\beta}}(\sigma) &:= e^{-is\langle\xi+\sigma\rangle} \widehat{P_{k_2}w}(s, \xi + \sigma), \quad |\sigma| \sim 2^{l_2}, \\ \widehat{\tilde{\gamma}}(\zeta) &:= e^{is\langle-\xi+\zeta\rangle} \widehat{P_{k_3}w}(s, -\xi + \zeta)\end{aligned}$$

to obtain

$$\begin{aligned}|J_{l_1 l_2, 3}| &\lesssim \|\mathcal{F}^{-1}(\partial_\eta m_1)\|_{L^1} \|\tilde{\alpha}\|_{L^2} \|\tilde{\beta}\|_{L^2} \|\tilde{\gamma}\|_{L^\infty} \\ &\lesssim 2^{-m} 2^{-l_1} 2^{-l_2} 2^{8k_+} 2^{2k} \cdot \epsilon_1 2^{l_1/2} 2^{-(N_1+10)k_+} \cdot \epsilon_1 2^{l_2/2} 2^{-(N_1+10)k_+} \cdot \epsilon_1 2^{-m/2} \\ &\lesssim \epsilon_1^3 2^{-m} 2^{-m/2} 2^{-l_1/2} 2^{-l_2/2} 2^{-(N_1+10)k_+} \\ &\lesssim \epsilon_1^3 2^{-m} 2^{-m/20} 2^{-(N_1+10)k_+}.\end{aligned}$$

Therefore, the estimate (4.67) is established.

**Case 2b:**  $\eta$  is away from the space-time resonance set. In this case, applying similar argument as above, we can prove that

$$|J_{l_1 l_2}(s, \xi)| \lesssim \epsilon_1^3 2^{-m} 2^{-\delta_1 m} 2^{-(N_1+10)k_+}, \quad l_1 \geq \max(l_2, \bar{l} + 1). \quad (4.70)$$

Further details are omitted here since the proof is almost the same as Case 2a.

**Case 2c:**  $(\eta, \sigma)$  is near the space-time resonance set. In this case, the above strategy is not workable as both  $\eta$  and  $\sigma$  can be very small, and a phase correction is needed to close the argument. Our aim is to show

$$\left| J_{\bar{l}}(s, \xi) - 2\pi \frac{c^*(\xi)\langle\xi\rangle^3 \widehat{P_{k_1}w}(s, \xi) \widehat{P_{k_2}w}(s, \xi) \widehat{P_{k_3}\bar{w}}(s, -\xi)}{s+1} \right| \lesssim \epsilon_1^3 2^{-m} 2^{-\delta_1 m} 2^{-(N_1+10)k_+} \quad (4.71)$$

for some  $\delta_1 > 0$ . To prove (4.71), we use

$$\begin{aligned} \text{LHS of (4.71)} &\leq |J_{\bar{l}}(s, \xi) - \tilde{J}_{\bar{l}}(s, \xi)| + |\tilde{J}_{\bar{l}}(s, \xi) - \bar{J}_{\bar{l}}(s, \xi)| \\ &\quad + \left| \bar{J}_{\bar{l}}(s, \xi) - 2\pi \frac{c^*(\xi)\langle\xi\rangle^3 \widehat{P_{k_1}w}(s, \xi) \widehat{P_{k_2}w}(s, \xi) \widehat{P_{k_3}\bar{w}}(s, -\xi)}{s+1} \right|, \end{aligned}$$

where

$$\begin{aligned} \tilde{J}_{\bar{l}}(s, \xi) &:= \int_{\mathbb{R}^2} \tilde{c}_{k_1 k_2 k_3}^{++-}(\xi, \eta, \sigma) e^{\frac{is\eta\sigma}{\langle\xi\rangle^3}} \\ &\quad \cdot \widehat{P_{k_1}w}(\xi + \eta) \widehat{P_{k_2}w}(\xi + \sigma) \widehat{P_{k_3}\bar{w}}(-\xi - \eta - \sigma) \varphi(2^{-\bar{l}}\eta) \varphi(2^{-\bar{l}}\sigma) d\eta d\sigma, \\ \bar{J}_{\bar{l}}(s, \xi) &:= \int_{\mathbb{R}^2} c^*(\xi) e^{\frac{is\eta\sigma}{\langle\xi\rangle^3}} \widehat{P_{k_1}w}(\xi) \widehat{P_{k_2}w}(\xi) \widehat{P_{k_3}\bar{w}}(-\xi) \varphi(2^{-\bar{l}}\eta) \varphi(2^{-\bar{l}}\sigma) d\eta d\sigma. \end{aligned}$$

By using Taylor's expansion, we have

$$\begin{aligned} \Psi(\xi, \eta, \sigma) &= \Psi(\xi, 0, 0) + \Psi_\eta(\xi, 0, 0)\eta + \Psi_\sigma(\xi, 0, 0)\sigma \\ &\quad + \frac{1}{2}\Psi_{\eta\eta}(\xi, 0, 0)\eta^2 + \frac{1}{2}\Psi_{\sigma\sigma}(\xi, 0, 0)\sigma^2 + \Psi_{\eta\sigma}(\xi, 0, 0)\eta\sigma + \text{remainder}, \end{aligned}$$

which implies, by (4.64) and the fact  $|\eta|, |\sigma| \sim 2^{\bar{l}}$ ,

$$|\Psi(\xi, \eta, \sigma) - \langle\xi\rangle^{-3}\eta\sigma| \lesssim 2^{-4k_+}(|\eta| + |\sigma|)^3 \lesssim 2^{-4k_+} 2^{3\bar{l}}.$$

Combining (4.40), (4.63) and (4.43) yields ( $2^{\bar{l}} \sim 2^{-9m/20}$ )

$$\begin{aligned} \left| J_{\bar{l}}(s, \xi) - \tilde{J}_{\bar{l}}(s, \xi) \right| &\lesssim \int_{\mathbb{R}^2} |\tilde{c}_{k_1 k_2 k_3}^{++-}(\xi, \eta, \sigma)| \cdot |e^{is\Psi} - e^{\frac{is\eta\sigma}{\langle\xi\rangle^3}}| \\ &\quad \cdot |\widehat{P_{k_1}w}(\xi + \eta) \widehat{P_{k_2}w}(\xi + \sigma) \widehat{P_{k_3}\bar{w}}(-\xi - \eta - \sigma)| d\eta d\sigma \\ &\lesssim 2^{5k_+} \cdot 2^m 2^{-4k_+} 2^{3\bar{l}} \cdot \epsilon_1^3 2^{-3(N_1+10)k_+} \cdot 2^{2\bar{l}} \\ &\lesssim \epsilon_1^3 2^{-5m/4} 2^{-(N_1+10)k_+}. \end{aligned} \quad (4.72)$$

In order to estimate the term  $\tilde{J}_{\bar{l}}(s, \xi) - \bar{J}_{\bar{l}}(s, \xi)$ , note that

$$|\tilde{c}^{++-}(\xi, \eta, \sigma) - c^*(\xi)| = |\tilde{c}^{++-}(\xi, \eta, \sigma) - \tilde{c}^{++-}(\xi, 0, 0)| \lesssim 2^{\bar{l}} 2^{5k_+},$$

and by (4.44),

$$|\widehat{P_{k_1}w}(s, \xi + \zeta) - \widehat{P_{k_1}w}(s, \xi)| \lesssim \|\partial_\xi \widehat{P_{k_1}w}\|_{L^2} 2^{\bar{l}/2} \lesssim \epsilon_1 2^{p_0 m} 2^{-(N_1-4)k_+} 2^{\bar{l}/2}, \quad |\zeta| \lesssim 2^{\bar{l}}.$$

So it is easy to see

$$\begin{aligned} & \left| \tilde{c}^{++-}(\xi, \eta, \sigma) \widehat{P_{k_1}w}(s, \xi + \eta) \widehat{P_{k_2}w}(s, \xi + \sigma) \widehat{P_{k_3}w}(s, -\xi - \eta - \sigma) \right. \\ & \quad \left. - c^*(\xi) \widehat{P_{k_1}w}(s, \xi) \widehat{P_{k_2}w}(s, \xi) \widehat{P_{k_3}w}(s, -\xi) \right| \\ & \lesssim \epsilon_1^3 2^{\bar{l}} 2^{-(3N_1+25)k_+} + \epsilon_1^3 2^{\bar{l}/2} 2^{p_0 m} 2^{-(3N_1+11)k_+} \end{aligned}$$

whenever  $|\eta|, |\sigma| \lesssim 2^{\bar{l}}$ , where we have used (4.43) in the above estimate. Therefore

$$\begin{aligned} |\tilde{J}_{\bar{l}}(s, \xi) - \bar{J}_{\bar{l}}(s, \xi)| & \lesssim \epsilon_1^3 2^{3\bar{l}} 2^{-(3N_1+25)k_+} + \epsilon_1^3 2^{5\bar{l}/2} 2^{p_0 m} 2^{-(3N_1+11)k_+} \\ & \lesssim \epsilon_1^3 2^{-m} 2^{-m/8} 2^{p_0 m} 2^{-(N_1+10)k_+}. \end{aligned} \quad (4.73)$$

Now, using (4.43) and applying Lemma 4.13 with  $\lambda = s/\langle \xi \rangle^3$ ,  $\mu = 2^{\bar{l}}$  and  $n = 1$ , we have

$$\begin{aligned} & \left| \bar{J}_{\bar{l}}(s, \xi) - 2\pi \frac{c^*(\xi) \langle \xi \rangle^3 \widehat{P_{k_1}w}(s, \xi) \widehat{P_{k_2}w}(s, \xi) \widehat{P_{k_3}w}(s, -\xi)}{s+1} \right| \\ & \lesssim \left| \bar{J}_{\bar{l}}(s, \xi) - 2\pi \frac{c^*(\xi) \langle \xi \rangle^3 \widehat{P_{k_1}w}(s, \xi) \widehat{P_{k_2}w}(s, \xi) \widehat{P_{k_3}w}(s, -\xi)}{s} \right| \\ & \quad + |c^*(\xi) \langle \xi \rangle^3 \widehat{P_{k_1}w}(s, \xi) \widehat{P_{k_2}w}(s, \xi) \widehat{P_{k_3}w}(s, -\xi)| \cdot \left( \frac{1}{s} - \frac{1}{s+1} \right) \\ & \lesssim \epsilon_1^3 2^{2k} 2^{-(3N_1+27)k_+} \left| \int_{\mathbb{R}^2} e^{\frac{is\eta\sigma}{\langle \xi \rangle^3}} \varphi(2^{-\bar{l}}\eta) \varphi(2^{-\bar{l}}\sigma) d\eta d\sigma - 2\pi \langle \xi \rangle^3 s^{-1} \right| \\ & \quad + \epsilon_1^3 2^{2k} 2^{-(3N_1+24)k_+} 2^{-2m} \\ & \lesssim \epsilon_1^3 2^{-(3N_1+25)k_+} 2^{-2m} 2^{6k_+} 2^{-2\bar{l}} + \epsilon_1^3 2^{-(3N_1+22)k_+} 2^{-2m} \\ & \lesssim \epsilon_1^3 2^{-m} 2^{-m/10} 2^{-(N_1+10)k_+}. \end{aligned} \quad (4.74)$$

Therefore, (4.71) follows from (4.72)–(4.74). This ends the proof of the lemma.  $\square$

**Lemma 4.15** *The estimate (4.52) holds under the conditions (4.56), (4.57) and*

$$\max(|k_1 - k|, |k_2 - k|, |k_3 - k|) \geq 21, \quad \max(|k_1 - k_3|, |k_2 - k_3|) \geq 6. \quad (4.75)$$

*Proof.* Recall that

$$I_{k_1 k_2 k_3}^{++-} = \int_{\mathbb{R}^2} \tilde{c}_{k_1 k_2 k_3}^{++-}(\xi, \eta, \sigma) e^{is\Psi(\xi, \eta, \sigma)} \widehat{P_{k_1}w}(\xi + \eta) \widehat{P_{k_2}w}(\xi + \sigma) \widehat{P_{k_3}w}(-\xi - \eta - \sigma) d\eta d\sigma,$$

where

$$\Psi(\xi, \eta, \sigma) = \langle \xi \rangle - \langle \xi + \eta \rangle - \langle \xi + \sigma \rangle + \langle \xi + \eta + \sigma \rangle,$$

and our aim is to show that there exists  $\delta_2 > 0$  such that

$$|I_{k_1 k_2 k_3}^{++-}(s, \xi)| \lesssim \epsilon_1^3 2^{-m} 2^{-\delta_2 m} 2^{-(N_1 + 10)k_+}. \quad (4.76)$$

According to (4.75), we may assume  $|k_1 - k_3| \geq 6$ . Since  $-\sigma = (\xi + \eta) + (-\xi - \eta - \sigma)$ , then we have  $|\sigma| \sim \max(|\xi + \eta|, |\xi + \eta + \sigma|) = 2^{\max(k_1, k_3)}$  and

$$|\partial_\eta \Psi| = \left| -\frac{\xi + \eta}{\langle \xi + \eta \rangle} + \frac{\xi + \eta + \sigma}{\langle \xi + \eta + \sigma \rangle} \right| \gtrsim 2^{-3 \max(k_1, k_3)} 2^{\max(k_1, k_3)}, \quad (4.77)$$

$$|\partial_\eta^2 \Psi| = \left| -\frac{1}{\langle \xi + \eta \rangle^3} + \frac{1}{\langle \xi + \eta + \sigma \rangle^3} \right| \lesssim 2^{-5 \min(k_1, k_3)} 2^{2 \max(k_1, k_3)}. \quad (4.78)$$

Integration by parts with respect to  $\eta$  gives

$$|I_{k_1 k_2 k_3}^{++-}(s, \xi)| \leq |F_1(s, \xi)| + |F_2(s, \xi)| + |F_3(s, \xi)|,$$

where

$$\begin{aligned} F_1(s, \xi) &:= \int_{\mathbb{R}^2} e^{is\Psi} m_2 \partial_\eta \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \widehat{P_{k_3} \bar{w}}(-\xi - \eta - \sigma) d\eta d\sigma, \\ F_2(s, \xi) &:= \int_{\mathbb{R}^2} e^{is\Psi} m_2 \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \partial_\eta \widehat{P_{k_3} \bar{w}}(-\xi - \eta - \sigma) d\eta d\sigma, \\ F_3(s, \xi) &:= \int_{\mathbb{R}^2} e^{is\Psi} \partial_\eta m_2 \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \widehat{P_{k_3} \bar{w}}(-\xi - \eta - \sigma) d\eta d\sigma \end{aligned}$$

with

$$m_2 = m_2(\eta, \sigma) := (s \partial_\eta \Psi)^{-1} \cdot \tilde{c}_{k_1 k_2 k_3}^{++-}.$$

Using the bounds (4.40), (4.77), (4.78) and Lemma 4.12, we can obtain

$$\|\mathcal{F}^{-1} m_2\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-m} 2^{10 \max(k_1, k_3)} 2^{-\max(k_1, k_3)}, \quad (4.79)$$

$$\|\mathcal{F}^{-1}(\partial_\eta m_2)\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-m} 2^{13 \max(k_1, k_3)} 2^{-5 \min(k_1, k_3)}. \quad (4.80)$$

Applying Lemma 4.11 with

$$\begin{aligned} \widehat{\alpha}(\eta) &:= e^{-is\langle \xi + \eta \rangle} \partial_\eta \widehat{P_{k_1} w}(s, \xi + \eta), \\ \widehat{\beta}(\sigma) &:= e^{-is\langle \xi + \sigma \rangle} \widehat{P_{k_2} w}(s, \xi + \sigma), \\ \widehat{\gamma}(\zeta) &:= e^{is\langle -\xi + \zeta \rangle} \widehat{P_{k_3} \bar{w}}(s, -\xi + \zeta), \end{aligned}$$

we use (4.79), (4.44), (4.48) and (4.46) to get

$$|F_1(s, \xi)| \lesssim \|\mathcal{F}^{-1} m_2\|_{L^1} \|\alpha\|_{L^2} \|\beta\|_{L^2} \|\gamma\|_{L^\infty}$$

$$\begin{aligned}
&\lesssim 2^{-m} 2^{10 \max(k_1, k_3)} 2^{-\max(k_1, k_3)} \cdot \epsilon_1 2^{p_0 m} 2^{-(N_1-4)k_1+} \\
&\quad \cdot \epsilon_1 2^{\max(k_1, k_3)/2} 2^{-(N_1+10)k_2+} \cdot \epsilon_1 2^{-m/2} \\
&\lesssim \epsilon_1^3 2^{-3m/2} 2^{p_0 m} 2^{-(N_1+10)k_+} 2^{(N_1+10)k_+} 2^{-\max(k_1, k_3)/2}.
\end{aligned}$$

Notice that  $k \leq \max(k_1, k_2, k_3) + 2$ , so the assumptions (4.56)–(4.57) yield

$$(N_1 + 10) \max(k_1, k_2, k_3) \leq m/6, \quad \text{med}(k_1, k_2, k_3) \geq -3m/5.$$

Since

$$\begin{aligned}
\max(k_1, k_3) &= \text{med}(k_1, k_2, k_3), \quad \text{if } k_2 = \max(k_1, k_2, k_3), \\
\max(k_1, k_3) &\geq \text{med}(k_1, k_2, k_3), \quad \text{if } k_2 < \max(k_1, k_2, k_3),
\end{aligned}$$

we also get  $\max(k_1, k_3) \geq -3m/5$ . Therefore, we conclude

$$\begin{aligned}
|F_1(s, \xi)| &\lesssim \epsilon_1^3 2^{-3m/2} 2^{p_0 m} 2^{-(N_1+10)k_+} 2^{(N_1+10)\max(k_1, k_2, k_3)+} 2^{-\max(k_1, k_3)/2} \\
&\lesssim \epsilon_1^3 2^{-m} 2^{p_0 m} 2^{-m/30} 2^{-(N_1+10)k_+}.
\end{aligned}$$

With the same treatment, we can get the same bound for  $|F_2(s, \xi)|$ . Finally, using Lemma 4.11, (4.80), (4.45) and (4.46), we can obtain

$$\begin{aligned}
|F_3(s, \xi)| &\lesssim 2^{-m} 2^{13 \max(k_1, k_3)} 2^{-5 \min(k_1, k_3)+} \cdot \epsilon_1^2 2^{2p_0 m} 2^{-(N-5)\max(k_1, k_2, k_3)+} \cdot \epsilon_1 2^{-m/2} \\
&\lesssim \epsilon_1^3 2^{-m} 2^{2p_0 m} 2^{-m/2} 2^{-(N_1+10)k_+}.
\end{aligned}$$

By combining the estimates for  $F_1$ ,  $F_2$  and  $F_3$ , we deduce the desired bound (4.76).  $\square$

**Lemma 4.16** *The estimate (4.52) holds under the hypotheses (4.56), (4.57) and*

$$\max(|k_1 - k|, |k_2 - k|, |k_3 - k|) \geq 21, \quad \max(|k_1 - k_3|, |k_2 - k_3|) \leq 5. \quad (4.81)$$

*Proof.* Recall that we want to show

$$|I_{k_1 k_2 k_3}^{++-}(s, \xi)| \lesssim \epsilon_1^3 2^{-m} 2^{-\delta_3 m} 2^{-(N_1+10)k_+} \quad (4.82)$$

for some  $\delta_3 > 0$ , where the definition of  $I_{k_1 k_2 k_3}^{++-}$  is the same as in Lemma 4.15. According to (4.81), we may assume  $k_1, k_2, k_3 \geq k + 11$ , then it follows from (4.57) that

$$2^{k_1} \sim 2^{k_2} \sim 2^{k_3} \gtrsim 2^{-3m/5}. \quad (4.83)$$

Since  $\eta = (\xi + \eta) - \xi$  and  $\sigma = (\xi + \sigma) - \xi$ , we also have  $|\eta| \sim |\sigma| \sim 2^{k_1}$ . Therefore,

$$\begin{aligned}
|\partial_\eta \Psi| &= \left| -\frac{\xi + \eta}{\langle \xi + \eta \rangle} + \frac{\xi + \eta + \sigma}{\langle \xi + \eta + \sigma \rangle} \right| \sim 2^{-3k_1+} 2^{k_1}, \\
|\partial_\eta^2 \Psi| &= \left| -\frac{1}{\langle \xi + \eta \rangle^3} + \frac{1}{\langle \xi + \eta + \sigma \rangle^3} \right| \sim 2^{-5k_1+} 2^{2k_1}.
\end{aligned}$$

Now, with integration by parts in  $\eta$ , we see

$$|I_{k_1 k_2 k_3}^{++-}(s, \xi)| \leq |G_1(s, \xi)| + |G_2(s, \xi)| + |G_3(s, \xi)|,$$

where

$$\begin{aligned} G_1(s, \xi) &:= \int_{\mathbb{R}^2} e^{is\Psi} m_3 \partial_\eta \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \widehat{P_{k_3} \bar{w}}(-\xi - \eta - \sigma) d\eta d\sigma, \\ G_2(s, \xi) &:= \int_{\mathbb{R}^2} e^{is\Psi} m_3 \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \partial_\eta P_{k_3} \bar{w}(-\xi - \eta - \sigma) d\eta d\sigma, \\ G_3(s, \xi) &:= \int_{\mathbb{R}^2} e^{is\Psi} \partial_\eta m_3 \widehat{P_{k_1} w}(\xi + \eta) \widehat{P_{k_2} w}(\xi + \sigma) \widehat{P_{k_3} \bar{w}}(-\xi - \eta - \sigma) d\eta d\sigma \end{aligned}$$

with

$$m_3 = m_3(\eta, \sigma) := (s \partial_\eta \Psi)^{-1} \cdot \tilde{c}_{k_1 k_2 k_3}^{++-}.$$

From (4.40), Lemma 4.12 and the bounds for  $\partial_\eta \Psi$  and  $\partial_\eta^2 \Psi$ , it is easy to see

$$\|\mathcal{F}^{-1} m_3\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-m} 2^{10k_1+} 2^{-k_1}, \quad \|\mathcal{F}^{-1} (\partial_\eta m_2)\|_{L^1(\mathbb{R}^2)} \lesssim 2^{-m} 2^{8k_1+}.$$

Applying Lemma 4.11 with

$$\begin{aligned} \widehat{\alpha}(\eta) &:= e^{-is\langle \xi + \eta \rangle} \partial_\eta \widehat{P_{k_1} w}(s, \xi + \eta), \\ \widehat{\beta}(\sigma) &:= e^{-is\langle \xi + \sigma \rangle} \widehat{P_{k_2} w}(s, \xi + \sigma), \\ \widehat{\gamma}(\zeta) &:= e^{is\langle -\xi + \zeta \rangle} \widehat{P_{k_3} \bar{w}}(s, -\xi + \zeta), \end{aligned}$$

and using (4.83), we deduce

$$\begin{aligned} |G_1(s, \xi)| &\lesssim \|\mathcal{F}^{-1} m_3\|_{L^1(\mathbb{R}^2)} \|\alpha\|_{L^2} \|\beta\|_{L^2} \|\gamma\|_{L^\infty} \\ &\lesssim 2^{-m} 2^{10k_1+} 2^{-k_1} \cdot \epsilon_1 2^{p_0 m} 2^{-(N_1-4)k_1+} \cdot \epsilon_1 2^{k_2/2} 2^{-(N_1+10)k_2+} \cdot \epsilon_1 2^{-m/2} \\ &\lesssim \epsilon_1^3 2^{-m} 2^{p_0 m} 2^{-m/30} 2^{-(N_1+10)k_+}. \end{aligned}$$

Similarly, we can obtain the same bound for the term  $|G_2(s, \xi)|$ . To estimate  $|G_3(s, \xi)|$ , we again use Lemma 4.11 to get

$$\begin{aligned} |G_3(s, \xi)| &\lesssim 2^{-m} 2^{8k_1+} \cdot \epsilon_1 2^{p_0 m} 2^{-(N-5)k_1+} \cdot \epsilon_1 2^{p_0 m} 2^{-(N-5)k_2+} \cdot \epsilon_1 2^{-m/2} \\ &\lesssim \epsilon_1^3 2^{-3m/2} 2^{2p_0 m} 2^{-(N_1+10)k_+}. \end{aligned}$$

The proof of Lemma 4.16 is completed.  $\square$

**Lemma 4.17** *The estimate (4.53) holds under the assumptions of Proposition 4.9.*

*Proof.* By a simple change of variables, we rewrite the LHS of (4.53) as

$$\int_{t_1}^{t_2} e^{i\vartheta(s, \xi)} I_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(s, \xi) ds$$

$$= \int_{t_1}^{t_2} e^{i\vartheta} \left[ \int_{\mathbb{R}^2} \tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3} e^{is\Psi^{\iota_1 \iota_2 \iota_3}} \widehat{P_{k_1} w^{\iota_1}}(\xi + \eta) \widehat{P_{k_2} w^{\iota_2}}(\xi + \sigma) \widehat{P_{k_3} w^{\iota_3}}(-\xi - \eta - \sigma) d\eta d\sigma \right] ds,$$

where

$$\begin{aligned} \tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma) &:= c_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(\xi + \eta, \xi + \sigma, -\xi - \eta - \sigma), \\ \Psi^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma) &= \langle \xi \rangle - \iota_1 \langle \xi + \eta \rangle - \iota_2 \langle \xi + \sigma \rangle - \iota_3 \langle \xi + \eta + \sigma \rangle. \end{aligned}$$

Note that the phase  $\Psi^{\iota_1 \iota_2 \iota_3}(\xi, \eta, \sigma)$  never vanishes when  $\iota_1 \iota_2 \iota_3 \in \{+--, ++-, ---\}$ . So we use integration by parts in  $s$  to obtain

$$\int_{t_1}^{t_2} e^{iH(s, \xi)} I_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}(s, \xi) ds := K_1(t_1, \xi) + K_2(t_2, \xi) + L_1(\xi) + L_2(\xi),$$

where

$$K_1(t_1, \xi) :=$$

$$- e^{i\vartheta(t_1, \xi)} \int_{\mathbb{R}^2} \frac{\tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3} e^{it_1 \Psi^{\iota_1 \iota_2 \iota_3}}}{i\Psi^{\iota_1 \iota_2 \iota_3}} \widehat{P_{k_1} w^{\iota_1}}(t_1, \xi + \eta) \widehat{P_{k_2} w^{\iota_2}}(t_1, \xi + \sigma) \widehat{P_{k_3} w^{\iota_3}}(t_1, -\xi - \eta - \sigma) d\eta d\sigma,$$

$$K_2(t_2, \xi) :=$$

$$e^{i\vartheta(t_2, \xi)} \int_{\mathbb{R}^2} \frac{\tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3} e^{it_2 \Psi^{\iota_1 \iota_2 \iota_3}}}{i\Psi^{\iota_1 \iota_2 \iota_3}} \widehat{P_{k_1} w^{\iota_1}}(t_2, \xi + \eta) \widehat{P_{k_2} w^{\iota_2}}(t_2, \xi + \sigma) \widehat{P_{k_3} w^{\iota_3}}(t_2, -\xi - \eta - \sigma) d\eta d\sigma,$$

and

$$L_1(\xi) :=$$

$$- \int_{t_1}^{t_2} e^{i\vartheta} \vartheta_s \left[ \int_{\mathbb{R}^2} \frac{\tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3} e^{is\Psi^{\iota_1 \iota_2 \iota_3}}}{\Psi^{\iota_1 \iota_2 \iota_3}} \widehat{P_{k_1} w^{\iota_1}}(\xi + \eta) \widehat{P_{k_2} w^{\iota_2}}(\xi + \sigma) \widehat{P_{k_3} w^{\iota_3}}(-\xi - \eta - \sigma) d\eta d\sigma \right] ds,$$

$$L_2(\xi) :=$$

$$- \int_{t_1}^{t_2} e^{i\vartheta} \left[ \int_{\mathbb{R}^2} \frac{\tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3} e^{is\Psi^{\iota_1 \iota_2 \iota_3}}}{i\Psi^{\iota_1 \iota_2 \iota_3}} \partial_s \widehat{P_{k_1} w^{\iota_1}}(\xi + \eta) \widehat{P_{k_2} w^{\iota_2}}(\xi + \sigma) \widehat{P_{k_3} w^{\iota_3}}(-\xi - \eta - \sigma) d\eta d\sigma \right] ds$$

$$- \int_{t_1}^{t_2} e^{i\vartheta} \left[ \int_{\mathbb{R}^2} \frac{\tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3} e^{is\Psi^{\iota_1 \iota_2 \iota_3}}}{i\Psi^{\iota_1 \iota_2 \iota_3}} \widehat{P_{k_1} w^{\iota_1}}(\xi + \eta) \partial_s \widehat{P_{k_2} w^{\iota_2}}(\xi + \sigma) \widehat{P_{k_3} w^{\iota_3}}(-\xi - \eta - \sigma) d\eta d\sigma \right] ds$$

$$- \int_{t_1}^{t_2} e^{i\vartheta} \left[ \int_{\mathbb{R}^2} \frac{\tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3} e^{is\Psi^{\iota_1 \iota_2 \iota_3}}}{i\Psi^{\iota_1 \iota_2 \iota_3}} \widehat{P_{k_1} w^{\iota_1}}(\xi + \eta) \widehat{P_{k_2} w^{\iota_2}}(\xi + \sigma) \partial_s \widehat{P_{k_3} w^{\iota_3}}(-\xi - \eta - \sigma) d\eta d\sigma \right] ds.$$

Hence, in order to establish this lemma, it suffices to prove that there exists  $\delta_4 > 0$  such that

$$|K_1(t_1, \xi)| + |K_2(t_2, \xi)| + |L_1(\xi)| + |L_2(\xi)| \lesssim \epsilon_1^3 2^{-\delta_4 m} 2^{-(N_1 + 10)k_+} \quad (4.84)$$

whenever  $|\xi| \sim 2^k$  and  $\iota_1 \iota_2 \iota_3 \in \{+--, ++-, ---\}$ .

We first prove (4.84) for the case  $\iota_1 \iota_2 \iota_3 = +--$ . It is easy to see

$$|\Psi^{+--}|^{-1} \lesssim \langle \xi \rangle + \langle \xi + \eta \rangle + \langle \xi + \sigma \rangle + \langle \xi + \eta + \sigma \rangle \lesssim \langle \xi + \eta \rangle + \langle \xi + \sigma \rangle + \langle \xi + \eta + \sigma \rangle,$$

then using (4.40) and Lemma 4.12, we can see

$$\|\mathcal{F}^{-1}[(\Psi^{\iota_1 \iota_2 \iota_3})^{-1} \tilde{c}_{k_1 k_2 k_3}^{\iota_1 \iota_2 \iota_3}]\|_{L^1(\mathbb{R}^2)} \lesssim 2^{8 \max(k_1, k_2, k_3)_+}. \quad (4.85)$$

Applying Lemma 4.11 with

$$\begin{aligned}\widehat{\alpha}(\eta) &:= e^{-it_j \langle \xi + \eta \rangle} \widehat{P_{k_1} w}(t_j, \xi + \eta), \\ \widehat{\beta}(\sigma) &:= e^{it_j \langle \xi + \sigma \rangle} \widehat{P_{k_2} w}(t_j, \xi + \sigma), \\ \widehat{\gamma}(\zeta) &:= e^{it_j \langle -\xi + \zeta \rangle} \widehat{P_{k_3} w}(t_j, -\xi + \zeta),\end{aligned}$$

for  $j = 1, 2$ , and using (4.85), (4.45), (4.46), we can obtain, by estimating the lowest frequency component in  $L^\infty$  and the other two components in  $L^2$ ,

$$\begin{aligned}|K_1(t_1, \xi)| + |K_2(t_2, \xi)| &\lesssim 2^{8 \max(k_1, k_2, k_3)_+} \cdot \epsilon_1 2^{p_0 m} 2^{-(N-5) \max(k_1, k_2, k_3)_+} \cdot \epsilon_1 2^{p_0 m} 2^{-(N-5) \text{med}(k_1, k_2, k_3)_+} \cdot \epsilon_1 2^{-m/2} \\ &\lesssim \epsilon_1^3 2^{-m/2} 2^{2p_0 m} 2^{-(N_1+10)k_+}.\end{aligned}$$

To estimate  $L_1(\xi)$ , note that

$$|\vartheta_s(s, \xi)| \lesssim c^*(\xi) \langle \xi \rangle^3 (1+s)^{-1} |\widehat{w}(s, \xi)|^2 \lesssim \epsilon_1^2 2^{-m} 2^{2k} 2^{6k_+} 2^{-(2N_1+20)k_+}, \quad |\xi| \sim 2^k, \quad (4.86)$$

then using Lemma 4.11, (4.85), (4.86), (4.45) and (4.46), we obtain

$$\begin{aligned}|L_1(\xi)| &\lesssim 2^m \cdot \epsilon_1^2 2^{-m} 2^{-(2N_1+12)k_+} \cdot 2^{8 \max(k_1, k_2, k_3)_+} \cdot \epsilon_1 2^{p_0 m} 2^{-(N-5) \max(k_1, k_2, k_3)_+} \\ &\quad \cdot \epsilon_1 2^{p_0 m} 2^{-(N-5) \text{med}(k_1, k_2, k_3)_+} \cdot \epsilon_1 2^{-m/2} \\ &\lesssim \epsilon_1^5 2^{-m/2} 2^{2p_0 m} 2^{-(N_1+10)k_+}.\end{aligned}$$

For the term  $L_2(\xi)$ , we use (4.45), (4.49) to get

$$\begin{aligned}|L_2(\xi)| &\lesssim 2^m \cdot 2^{8 \max(k_1, k_2, k_3)_+} \cdot \epsilon_1^5 2^{3p_0 m} 2^{-3m/2} \cdot (2^{-(N-7) \max(k_1, k_2, k_3)_+} 2^{-(N-7) \text{med}(k_1, k_2, k_3)_+} \\ &\quad + 2^{-(N-7) \min(k_1, k_2, k_3)_+} 2^{-(N-7) \max(k_1, k_2, k_3)_+}) \\ &\lesssim \epsilon_1^5 2^{-m/2} 2^{3p_0 m} 2^{-(N_1+10)k_+}.\end{aligned}$$

Therefore, the estimate (4.84) is established for  $\iota_1 \iota_2 \iota_3 = +--$ .

Note that

$$\begin{aligned}|\Psi^{+++}(\xi, \eta, \sigma)| &\geq \frac{2}{\langle \xi \rangle + \langle \xi + \eta \rangle + \langle \xi + \sigma \rangle + \langle \xi + \eta + \sigma \rangle}, \\ |\Psi^{---}(\xi, \eta, \sigma)| &\sim \max(\langle \xi \rangle, \langle \xi + \eta \rangle, \langle \xi + \sigma \rangle, \langle \xi + \eta + \sigma \rangle).\end{aligned}$$

Then we can apply the same argument as above to show the bound (4.84) in the case  $\iota_1 \iota_2 \iota_3 = +++$  and  $---$ . For the sake of simplicity, we omit further details. This ends the proof of the lemma.  $\square$

To complete the proof of Proposition 4.5, we are left to prove (4.58). That is, we are aiming to show

$$\left| \langle \xi \rangle^{N_1+10} \int_{t_1}^{t_2} e^{i\vartheta(s,\xi)} e^{is\langle \xi \rangle} \widehat{\mathcal{N}_R}(s, \xi) ds \right| \lesssim \epsilon_1^4 (1+t_1)^{-\delta}$$

for some  $\delta > 0$ . Recall that the definitions of  $\vartheta$  and  $\mathcal{N}_R$  are given by (4.37) and (4.30), respectively. To prove this bound, we need the following lemma.

**Lemma 4.18** *For any  $t \in [0, T]$ , there hold that*

$$\|\langle \xi \rangle^{N-20} \widehat{\mathcal{N}_R}(\xi)\|_{L^\infty} \lesssim \epsilon_1^4 (1+t)^{2p_0-1}, \quad (4.87)$$

$$\|\mathcal{N}_R\|_{H^{N_1+10}} \lesssim \epsilon_1^4 (1+t)^{p_0-3/2}, \quad (4.88)$$

$$\|\Gamma \mathcal{N}_R\|_{H^{N_1-10}} \lesssim \epsilon_1^4 (1+t)^{p_0-3/2}, \quad (4.89)$$

$$\|x \mathcal{N}_R\|_{H^{N_1-10}} \lesssim \epsilon_1^4 (1+t)^{p_0-1/2}. \quad (4.90)$$

Recall the bounds for  $g$  and  $h$

$$\begin{aligned} \|g\|_{H^{N-5}} + \|h\|_{H^N} &\lesssim \epsilon_1 (1+t)^{p_0}, \\ \|\Gamma g\|_{H^{N_1-5}} + \|\Gamma h\|_{H^{N_1}} &\lesssim \epsilon_1 (1+t)^{p_0}, \\ \|g\|_{W^{N_1+10,\infty}} + \|h\|_{W^{N_1+10,\infty}} &\lesssim \epsilon_1 (1+t)^{-1/2}, \end{aligned} \quad (4.91)$$

and the bounds for the difference  $h - g$

$$\begin{aligned} \|h - g\|_{H^{N-5}} &\lesssim \epsilon_1^2 (1+t)^{p_0-1/2}, \\ \|h - g\|_{W^{N_1+5,\infty}} &\lesssim \epsilon_1^2 (1+t)^{-1}, \\ \|\Gamma(h - g)\|_{H^{N_1-5}} &\lesssim \epsilon_1^2 (1+t)^{p_0-1/2}. \end{aligned} \quad (4.92)$$

The bounds (4.91) and (4.92) follow easily from (4.10), (4.3) and Lemma 4.4.

*Proof of Lemma 4.18.* According to the definitions (4.6) and (4.30), we see that in order to prove Lemma 4.18, it suffices to show each term in  $\mathcal{N}_R$  satisfies (4.87)–(4.90). In this proof, we mainly concentrate on the term

$$\mathcal{N}_R^{++++} := \mathcal{O}[\mathcal{O}[h, q^{++}]h, b^{++}]h - \mathcal{O}[\mathcal{O}[g, q^{++}]g, b^{++}]g,$$

and the treatments for the other terms are similar. Decompose this term as

$$\mathcal{N}_R^{++++} = \mathcal{N}_{R1}^{++++} + \mathcal{N}_{R2}^{++++} + \mathcal{N}_{R3}^{++++}$$

with

$$\mathcal{N}_{R1}^{++++} := \mathcal{O}[\mathcal{O}[h, q^{++}]h, b^{++}](h - g),$$

$$\begin{aligned}\mathcal{N}_{R2}^{++++} &:= \mathcal{O}[\mathcal{O}[h-g, q^{++}]h, b^{++}]g, \\ \mathcal{N}_{R3}^{++++} &:= \mathcal{O}[\mathcal{O}[g, q^{++}](h-g), b^{++}]g.\end{aligned}$$

We first show (4.87). Using (4.22), (4.2), Lemma 4.3 and the bounds (4.91)–(4.92), we see

$$\begin{aligned}\|\langle \xi \rangle^{N-20} \mathcal{F} \mathcal{N}_{R1}^{++++}\|_{L^\infty} &\lesssim \|\mathcal{N}_{R1}^{++++}\|_{W^{N-20,1}} \lesssim \|\mathcal{O}[h, q^{++}]h\|_{H^{N-15}} \|h-g\|_{H^{N-15}} \\ &\lesssim \|h\|_{H^N} \|h\|_{L^\infty} \|h-g\|_{H^{N-5}} \lesssim \epsilon_1^4 (1+t)^{2p_0-1},\end{aligned}$$

and

$$\begin{aligned}\|\langle \xi \rangle^{N-20} \mathcal{F} \mathcal{N}_{R2}^{++++}\|_{L^\infty} &\lesssim \|\mathcal{O}[h-g, q^{++}]h\|_{H^{N-15}} \|g\|_{H^{N-15}} \\ &\lesssim (\|h-g\|_{H^{N-5}} \|h\|_{L^\infty} + \|h-g\|_{L^\infty} \|h\|_{H^N}) \|g\|_{H^{N-5}} \\ &\lesssim \epsilon_1^4 (1+t)^{2p_0-1}.\end{aligned}$$

The argument for the term  $\mathcal{N}_{R3}^{++++}$  is similar as above. Hence, the bound (4.87) follows.

Similarly, we have

$$\begin{aligned}\|\mathcal{N}_{R1}^{++++}\|_{H^{N_1+10}} &\lesssim \|\mathcal{O}[h, q^{++}]h\|_{H^{N_1+15}} \|h-g\|_{L^\infty} + \|\mathcal{O}[h, q^{++}]h\|_{L^\infty} \|h-g\|_{H^{N_1+15}} \\ &\lesssim \|h\|_{H^N} \|h\|_{L^\infty} \|h-g\|_{L^\infty} + \|h\|_{W^{4,\infty}}^2 \|h-g\|_{H^{N_1+15}} \\ &\lesssim \epsilon_1^4 (1+t)^{p_0-3/2},\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{N}_{R2}^{++++}\|_{H^{N_1+10}} &\lesssim \|\mathcal{O}[h-g, q^{++}]h\|_{H^{N_1+15}} \|g\|_{L^\infty} + \|\mathcal{O}[h-g, q^{++}]h\|_{L^\infty} \|g\|_{H^{N_1+15}} \\ &\lesssim (\|h-g\|_{H^{N-5}} \|h\|_{L^\infty} + \|h-g\|_{L^\infty} \|h\|_{H^N}) \|g\|_{L^\infty} \\ &\quad + \|h-g\|_{W^{4,\infty}} \|h\|_{W^{4,\infty}} \|g\|_{H^{N_1+15}} \\ &\lesssim \epsilon_1^4 (1+t)^{p_0-3/2}.\end{aligned}$$

Also, we can deal with the term  $\|\mathcal{N}_{R3}^{++++}\|_{H^{N_1+10}}$  in a similar way. Combining these estimates yields (4.88) as desired.

Now we prove the weighted estimate (4.89). As (4.24), we have

$$\Gamma \mathcal{N}_{R1}^{++++} = W_1 + W_2 + W_3$$

with

$$\begin{aligned}\widehat{W}_1(\xi) &:= \mathcal{F}(\mathcal{O}[\Gamma \mathcal{O}[h, q^{++}]h, b^{++}](h-g))(\xi), \\ \widehat{W}_2(\xi) &:= \mathcal{F}(\mathcal{O}[\mathcal{O}[h, q^{++}]h, b^{++}]\Gamma(h-g))(\xi), \\ \widehat{W}_3(\xi) &:= \frac{i}{2\pi} \int_{\mathbb{R}} \partial_\xi b^{++}(\xi-\eta, \eta) \mathcal{F}(\partial_t \mathcal{O}[h, q^{++}]h)(\xi-\eta) \widehat{(h-g)}(\eta) d\eta\end{aligned}$$

$$\begin{aligned}
& + \frac{i}{2\pi} \int_{\mathbb{R}} \partial_{\xi} b^{++}(\xi - \eta, \eta) \mathcal{F}(\mathcal{O}[h, q^{++}]h)(\xi - \eta) (\widehat{h-g})_t(\eta) d\eta \\
& + \frac{i}{2\pi} \int_{\mathbb{R}} \partial_{\eta} b^{++}(\xi - \eta, \eta) \mathcal{F}(\mathcal{O}[h, q^{++}]h)(\xi - \eta) (\widehat{h-g})_t(\eta) d\eta.
\end{aligned}$$

By expanding  $\Gamma\mathcal{O}[h, q^{++}]h$  as (4.24), we can obtain

$$\|\Gamma\mathcal{O}[h, q^{++}]h\|_{H^{N_1-5}} \lesssim \epsilon_1^2(1+t)^{p_0-1/2}.$$

Hence, using also (4.92), we have

$$\begin{aligned}
\|W_1\|_{H^{N_1-10}} & \lesssim \|\Gamma\mathcal{O}[h, q^{++}]h\|_{H^{N_1-5}} \|h-g\|_{L^\infty} + \|\Gamma\mathcal{O}[h, q^{++}]h\|_{L^2} \|h-g\|_{W^{N_1-5,\infty}} \\
& \lesssim \epsilon_1^4(1+t)^{p_0-3/2}.
\end{aligned}$$

Similarly, from (4.91)–(4.92), there holds

$$\begin{aligned}
\|W_2\|_{H^{N_1-10}} & \lesssim \|\mathcal{O}[h, q^{++}]h\|_{W^{N_1-5,\infty}} \|\Gamma(h-g)\|_{L^2} + \|\mathcal{O}[h, q^{++}]h\|_{L^\infty} \|\Gamma(h-g)\|_{H^{N_1-5}} \\
& \lesssim \epsilon_1^4(1+t)^{p_0-3/2}.
\end{aligned}$$

To estimate  $W_3$ , note that

$$\begin{aligned}
& \|h_t\|_{H^{N-5}} + \|g_t\|_{H^{N-10}} \lesssim \epsilon_1(1+t)^{p_0}, \\
& \|h_t\|_{W^{N_1,\infty}} + \|g_t\|_{W^{N_1,\infty}} \lesssim \epsilon_1(1+t)^{-1/2}, \\
& \|(h-g)_t\|_{L^\infty} \lesssim \|h_t\|_{L^\infty} \|h\|_{W^{5,\infty}} + \|h_t\|_{W^{5,\infty}} \|h\|_{L^\infty} \lesssim \epsilon_1^2(1+t)^{p_0-1}, \\
& \|(h-g)_t\|_{H^{N_1-5}} \lesssim \|h_t\|_{L^\infty} \|h\|_{H^{N_1}} + \|h_t\|_{H^{N_1}} \|h\|_{L^\infty} \lesssim \epsilon_1^2(1+t)^{p_0-1/2},
\end{aligned}$$

which can be verified by the equations (4.1), (4.5) and the identity (4.3), then

$$\begin{aligned}
\|W_3\|_{H^{N_1-10}} & \lesssim \|\partial_t \mathcal{O}[h, q^{++}]h\|_{H^{N_1-5}} \|h-g\|_{L^\infty} + \|\partial_t \mathcal{O}[h, q^{++}]h\|_{L^\infty} \|h-g\|_{H^{N_1-5}} \\
& \quad + \|\mathcal{O}[h, q^{++}]h\|_{H^{N_1-5}} \|(h-g)_t\|_{L^\infty} + \|\mathcal{O}[h, q^{++}]h\|_{L^\infty} \|(h-g)_t\|_{H^{N_1-5}} \\
& \lesssim \epsilon_1^4(1+t)^{p_0-3/2}.
\end{aligned}$$

Therefore, we conclude that

$$\|\Gamma\mathcal{N}_{R1}^{++++}\|_{H^{N_1-10}} \lesssim \|W_1\|_{H^{N_1-10}} + \|W_2\|_{H^{N_1-10}} + \|W_3\|_{H^{N_1-10}} \lesssim \epsilon_1^4(1+t)^{p_0-3/2}.$$

Moreover, we can estimate the  $H^{N_1-10}$  norm of  $\Gamma\mathcal{N}_{R2}^{++++}$  and  $\Gamma\mathcal{N}_{R3}^{++++}$  in a similar way as above, and we omit further details for simplicity. Thus, the bound (4.89) is valid.

Finally, by similar argument as the proof of (4.19), it is straightforward to obtain the desired bound for  $\|x\mathcal{N}_R\|_{H^{N_1-10}}$ . This ends the proof of the lemma.  $\square$

Now, we end this subsection by presenting the proof of Proposition 4.10.

*Proof of Proposition 4.10.* Denote

$$\widehat{\mathcal{A}}(\xi) := \int_{t_1}^{t_2} e^{i\vartheta(s,\xi)} e^{is\langle\xi\rangle} \widehat{\mathcal{N}_R}(s, \xi) ds = \widehat{\mathcal{A}}_1(\xi) + \widehat{\mathcal{A}}_2(\xi),$$

where

$$\begin{aligned}\widehat{\mathcal{A}}_1(\xi) &:= \int_{t_1}^{t_2} e^{i\vartheta(s,\xi)} e^{is\langle\xi\rangle} \left(1 - \varphi\left(\frac{\xi}{(1+s)^{p_0}}\right)\right) \widehat{\mathcal{N}_R}(s, \xi) ds, \\ \widehat{\mathcal{A}}_2(\xi) &:= \int_{t_1}^{t_2} e^{i\vartheta(s,\xi)} e^{is\langle\xi\rangle} \varphi\left(\frac{\xi}{(1+s)^{p_0}}\right) \widehat{\mathcal{N}_R}(s, \xi) ds.\end{aligned}$$

By (4.87) and the fact  $|\xi| \gtrsim (1+s)^{p_0}$ , it is easy to see

$$\begin{aligned}\|\langle\xi\rangle^{N_1+10} \widehat{\mathcal{A}}_1(\xi)\|_{L^\infty} &\lesssim \int_{t_1}^{t_2} \|\langle\xi\rangle^{N-20} \widehat{\mathcal{N}_R}(s, \xi)\|_{L^\infty} (1+s)^{-(N-N_1-30)p_0} ds \\ &\lesssim \int_{t_1}^{t_2} \epsilon_1^4 (1+s)^{2p_0-1} (1+s)^{-(N-N_1-30)p_0} ds \\ &\lesssim \epsilon_1^4 (1+t_1)^{-(N-N_1-32)p_0}.\end{aligned}\tag{4.93}$$

For the term  $\mathcal{A}_2$ , we use

$$\|\langle\xi\rangle^{N_1+10} \widehat{\mathcal{A}}_2(\xi)\|_{L^\infty} \lesssim \|\mathcal{A}_2\|_{H^{N_1+10}} + \|x\mathcal{A}_2\|_{H^{N_1+10}}.\tag{4.94}$$

The first term in the RHS of (4.94) can be estimated directly by (4.88)

$$\|\mathcal{A}_2\|_{H^{N_1+10}} \lesssim \int_{t_1}^{t_2} \|\mathcal{N}_R(s)\|_{H^{N_1+10}} ds \lesssim \int_{t_1}^{t_2} \epsilon_1^4 (1+s)^{p_0-3/2} ds \lesssim \epsilon_1^4 (1+t_1)^{p_0-1/2}.\tag{4.95}$$

To estimate the term  $\|x\mathcal{A}_2\|_{H^{N_1+10}}$ , we apply  $\partial_\xi$  to  $\widehat{\mathcal{A}}_2$ . In view of (4.26),

$$\begin{aligned}\langle\xi\rangle \partial_\xi (e^{it\langle\xi\rangle} \widehat{\mathcal{N}_R}) &= e^{it\langle\xi\rangle} \widehat{\Gamma \mathcal{N}_R} \\ &= e^{it\langle\xi\rangle} \widehat{\Gamma \mathcal{N}_R} - e^{it\langle\xi\rangle} [(\partial_t + i\langle\xi\rangle) \widehat{x \mathcal{N}_R}] \\ &= e^{it\langle\xi\rangle} \widehat{\Gamma \mathcal{N}_R} - \partial_t [e^{it\langle\xi\rangle} \widehat{x \mathcal{N}_R}],\end{aligned}$$

then  $\partial_\xi \widehat{\mathcal{A}}_2$  can be decomposed into

$$\begin{aligned}\partial_\xi \widehat{\mathcal{A}}_2(\xi) &= \int_{t_1}^{t_2} \partial_\xi [e^{i\vartheta(s,\xi)} \varphi\left(\frac{\xi}{(1+s)^{p_0}}\right)] e^{is\langle\xi\rangle} \widehat{\mathcal{N}_R}(s, \xi) ds \\ &\quad + \int_{t_1}^{t_2} e^{i\vartheta(s,\xi)} \varphi\left(\frac{\xi}{(1+s)^{p_0}}\right) \partial_\xi [e^{is\langle\xi\rangle} \widehat{\mathcal{N}_R}(s, \xi)] ds \\ &= \widehat{\mathcal{A}}_{21}(\xi) + \widehat{\mathcal{A}}_{22}(\xi) + \widehat{\mathcal{A}}_{23}(\xi).\end{aligned}$$

where

$$\widehat{\mathcal{A}}_{21}(\xi) := \int_{t_1}^{t_2} \partial_\xi [e^{i\vartheta(s,\xi)} \varphi\left(\frac{\xi}{(1+s)^{p_0}}\right)] e^{is\langle\xi\rangle} \widehat{\mathcal{N}_R}(s, \xi) ds,$$

$$\begin{aligned}\widehat{\mathcal{A}}_{22}(\xi) &:= \int_{t_1}^{t_2} e^{i\vartheta(s,\xi)} \langle \xi \rangle^{-1} \varphi\left(\frac{\xi}{(1+s)^{p_0}}\right) e^{is\langle \xi \rangle} \widehat{\Gamma\mathcal{N}_R}(s, \xi) ds, \\ \widehat{\mathcal{A}}_{23}(\xi) &:= - \int_{t_1}^{t_2} e^{i\vartheta(s,\xi)} \langle \xi \rangle^{-1} \varphi\left(\frac{\xi}{(1+s)^{p_0}}\right) \partial_s [e^{is\langle \xi \rangle} \widehat{x\mathcal{N}_R}(s, \xi)] ds.\end{aligned}$$

Using (4.37), (4.36), (4.50) and (4.10), we have

$$|\partial_\xi \vartheta(t, \xi)| \lesssim \langle \xi \rangle^8 \ln(1+t) \sup_{s \in [0,t]} (|\widehat{w}(s, \xi)|^2 + |\partial_\xi \widehat{w}(s, \xi)|^2) \lesssim \epsilon_1^2 (1+t)^{3p_0},$$

hence, by (4.88),

$$\|\mathcal{A}_{21}\|_{H^{N_1+10}} \lesssim \epsilon_1^4 \int_{t_1}^{t_2} (1+s)^{4p_0-3/2} ds \lesssim \epsilon_1^4 (1+t_1)^{4p_0-1/2}.$$

For the term  $\mathcal{A}_{22}$ , we obtain from (4.89) that

$$\begin{aligned}\|\mathcal{A}_{22}\|_{H^{N_1+10}} &\lesssim \int_{t_1}^{t_2} (1+s)^{19p_0} \|\Gamma\mathcal{N}_R\|_{H^{N_1-10}} ds \\ &\lesssim \epsilon_1^4 \int_{t_1}^{t_2} (1+s)^{20p_0-3/2} ds \lesssim \epsilon_1^4 (1+t_1)^{20p_0-1/2}.\end{aligned}$$

To estimate  $\|\mathcal{A}_{23}\|_{H^{N_1+10}}$ , using integration by parts in time, the bound

$$|\partial_s \vartheta(s, \xi)| \lesssim (1+s)^{-1} \langle \xi \rangle^8 |\widehat{w}(s, \xi)|^2 \lesssim \epsilon_1^2 (1+s)^{-1},$$

and (4.90), we obtain

$$\|\mathcal{A}_{23}\|_{H^{N_1+10}} \lesssim \epsilon_1^4 (1+t_1)^{20p_0-1/2}.$$

We finally conclude that

$$\|x\mathcal{A}_2\|_{H^{N_1+10}} \lesssim \|\mathcal{A}_{21}\|_{H^{N_1+10}} + \|\mathcal{A}_{22}\|_{H^{N_1+10}} + \|\mathcal{A}_{23}\|_{H^{N_1+10}} \lesssim \epsilon_1^4 (1+t_1)^{20p_0-1/2}. \quad (4.96)$$

Therefore, the desired bound (4.58) follows from (4.93)–(4.96).  $\square$

### 4.3 Proof of (4.13)

*Proof of (4.13).* Using Bernstein's inequality, (4.11) and (4.27), we have

$$\begin{aligned}\|xP_{\leqslant(1+t)^{1/240}} w\|_{H^{N_1+11}} &\lesssim (1+t)^{-1/240} \|P_{\leqslant(1+t)^{1/240}} w\|_{H^{N_1+11}} + \|P_{\leqslant(1+t)^{1/240}}(xw)\|_{H^{N_1+11}} \\ &\lesssim \|w\|_{H^{N-5}} + (1+t)^{1/16} \|P_{\leqslant(1+t)^{1/240}}(xw)\|_{H^{N_1-4}} \\ &\lesssim (\epsilon_0 + \epsilon_1^2)(1+t)^{1/16+p_0}.\end{aligned}$$

Then we deduce from the linear estimate (4.14), the bounds (4.12) and (4.27) that

$$\|P_{\leqslant(1+t)^{1/240}} g\|_{W^{N_1+10,\infty}} \lesssim (1+t)^{-1/2} \|\langle \xi \rangle^{N_1+10} \widehat{w}(\xi)\|_{L^\infty}$$

$$\begin{aligned}
& + (1+t)^{-5/8} (\|g\|_{H^{N_1+12}} + \|xP_{\leq(1+t)^{1/240}}w\|_{H^{N_1+11}}) \\
& \lesssim (1+t)^{-1/2} (\epsilon_0 + \epsilon_1^2), \quad \forall t \in [0, T].
\end{aligned} \tag{4.97}$$

On the other hand, by Bernstein's inequality and (4.27), there also holds

$$\begin{aligned}
\|P_{\geq(1+t)^{1/240}}g\|_{W^{N_1+10,\infty}} & \lesssim \|P_{\geq(1+t)^{1/240}}g\|_{H^{N_1+11}} \\
& \lesssim (1+t)^{-(N-N_1-11)/240} \|P_{\geq(1+t)^{1/240}}g\|_{H^N} \\
& \lesssim (1+t)^{-1/2} (\epsilon_0 + \epsilon_1^2).
\end{aligned} \tag{4.98}$$

Now, we conclude from (4.97) and (4.98) that

$$\|g(t)\|_{W^{N_1+10,\infty}} \lesssim \|P_{\leq(1+t)^{1/240}}g\|_{W^{N_1+10,\infty}} + \|P_{\geq(1+t)^{1/240}}g\|_{W^{N_1+10,\infty}} \lesssim (1+t)^{-1/2} (\epsilon_0 + \epsilon_1^2).$$

Moreover, if  $\epsilon_1$  is small enough, (4.3) and (4.17) lead to

$$\|h(t)\|_{W^{N_1+10,\infty}} \sim \|g(t)\|_{W^{N_1+10,\infty}}.$$

Therefore, (4.13) follows, and this also completes the proof of Proposition 4.1.  $\square$

Finally, combining Proposition 2.1, Propositions 3.1–3.2, Proposition 4.1 and Lemma 4.2, Theorem 1.1 follows by standard continuation argument.

## Appendix A

In this part, we prove the linear dispersive estimate for Klein-Gordon operator.

**Lemma A.1** *There holds that*

$$\|e^{\pm it\langle \partial_x \rangle} f\|_{L^\infty} \lesssim (1+t)^{-1/2} \|\widehat{f}\|_{L^\infty} + (1+t)^{-5/8} (\|f\|_{H^2} + \|xf\|_{H^1}), \quad \forall t \geq 0. \tag{A.1}$$

*Proof.* In this proof, we only show (A.1) in the “+” case, since the discussion for the minus case is similar. Note that the estimate (A.1) is trivial if  $0 \leq t \leq 100$ . Denote

$$J_k := \frac{1}{2\pi} \left| \int_{\mathbb{R}} e^{ix\xi + it\langle \xi \rangle} \widehat{P_k f}(\xi) d\xi \right|, \quad k \in \mathbb{Z},$$

then in order to prove this lemma, it suffices to show  $\sum_{k \in \mathbb{Z}} J_k \lesssim 1$  with  $f$  satisfying

$$t^{-1/2} \|\widehat{f}\|_{L^\infty} + t^{-5/8} (\|f\|_{H^2} + \|xf\|_{H^1}) \lesssim 1, \quad t > 100. \tag{A.2}$$

Now we divide this proof into four cases.

Case 1:  $2^k \leq t^{-1/2}$ . In this case, we use  $\|\widehat{P_k f}\|_{L^\infty} \lesssim t^{1/2}$  to obtain

$$\sum_{2^k \leq t^{-1/2}} J_k \lesssim \sum_{2^k \leq t^{-1/2}} 2^k \|\widehat{P_k f}\|_{L^\infty} \lesssim t^{1/2} \sum_{2^k \leq t^{-1/2}} 2^k \lesssim 1.$$

Case 2:  $2^k \geq t^{5/12}$ . By using the estimate  $2^{2k} \|\widehat{P_k f}\|_{L^2} \lesssim t^{5/8}$ , we have

$$\sum_{2^k \geq t^{5/12}} J_k \lesssim \sum_{2^k \geq t^{5/12}} 2^{k/2} \|\widehat{P_k f}\|_{L^2} \lesssim t^{5/8} \sum_{2^k \geq t^{5/12}} 2^{k/2} 2^{-2k} \lesssim 1.$$

Case 3:  $t^{-1/2} \leq 2^k \leq t^{5/12}$  and  $|x/t| \geq 1/2$ . Note that (A.2) implies

$$\|\widehat{P_k f}\|_{L^2} + 2^{2k} \|\widehat{P_k f}\|_{L^2} + \|\partial_\xi \widehat{P_k f}\|_{L^2} + 2^k \|\partial_\xi \widehat{P_k f}\|_{L^2} \lesssim t^{5/8}. \quad (\text{A.3})$$

Moreover, in this case we observe that

$$\|t f_x\|_{L^2} \leq 2 \|x f_x\|_{L^2} \lesssim t^{5/8} \Rightarrow \|f_x\|_{L^2} \lesssim t^{-3/8} \Rightarrow 2^k \|\widehat{P_k f}\|_{L^2} \lesssim t^{-3/8}. \quad (\text{A.4})$$

Subcase 3–1:  $|\xi| \geq 1/4$ . In this subcase, thanks to (A.4), it follows from the definition of  $J_k$  that

$$\sum_{4^{-1} \leq 2^k \leq t^{5/12}} J_k \lesssim \sum_{4^{-1} \leq 2^k} \|\widehat{P_k f}\|_{L^2} 2^{k/2} \lesssim \sum_{4^{-1} \leq 2^k} t^{-3/8} 2^{-k} 2^{k/2} \lesssim 1.$$

Subcase 3–2:  $|\xi| \leq 1/4$ . Let  $\Phi := x\xi + t\langle \xi \rangle$ , using integration by parts, we see that

$$\begin{aligned} J_k &= \left| \int_{\mathbb{R}} \partial_\xi (e^{i\Phi}) (i\partial_\xi \Phi)^{-1} \widehat{P_k f}(\xi) d\xi \right| \\ &\leq \left| \int_{\mathbb{R}} e^{i\Phi} (\partial_\xi \Phi)^{-2} \partial_\xi^2 \Phi \widehat{P_k f}(\xi) d\xi \right| + \left| \int_{\mathbb{R}} e^{i\Phi} (\partial_\xi \Phi)^{-1} \partial_\xi \widehat{P_k f}(\xi) d\xi \right|, \end{aligned} \quad (\text{A.5})$$

where  $\partial_\xi \Phi = t(xt^{-1} + \xi \langle \xi \rangle^{-1})$  and  $\partial_\xi^2 \Phi = t\langle \xi \rangle^{-3}$ . If  $|\xi| \leq 1/4$ , then  $|\partial_\xi^2 \Phi| \sim t$  and  $|\partial_\xi \Phi| \geq t(|x/t| - |\xi| \langle \xi \rangle^{-1}) \geq t/4$  since  $|\xi| \langle \xi \rangle^{-1} \leq 1/4$ . With the help of (A.3), it follows from (A.5) that

$$\sum_{t^{-1/2} \leq 2^k \leq 4^{-1}} J_k \lesssim \sum_{2^k \leq 4^{-1}} (t^{-1} \|\widehat{P_k f}\|_{L^2} 2^{k/2} + t^{-1} \|\partial_\xi \widehat{P_k f}\|_{L^2} 2^{k/2}) \lesssim t^{-1} t^{5/8} \lesssim 1.$$

Case 4:  $t^{-1/2} \leq 2^k \leq t^{5/12}$  and  $|x/t| \leq 1/2$ .

Subcase 4–1:  $|\xi| \geq 2$ . We see that  $|\partial_\xi^2 \Phi| \lesssim t|\xi|^{-3}$  and  $|\partial_\xi \Phi| \gtrsim t$ , so from (A.5), there holds

$$\sum_{2 \leq 2^k \leq t^{5/12}} J_k \lesssim \sum_{2 \leq 2^k} (t^{-1} 2^{-3k} \|\widehat{P_k f}\|_{L^2} 2^{k/2} + t^{-1} 2^{-k} 2^k \|\partial_\xi \widehat{P_k f}\|_{L^2} 2^{k/2}) \lesssim t^{-1} t^{5/8} \lesssim 1.$$

Subcase 4–2:  $|\xi| \leq 2$ . Let  $\xi_0$  be the unique root of the equation  $\partial_\xi \Phi = 0$ , i.e.,  $\xi_0 = -\frac{x}{\sqrt{t^2 - x^2}}$  and  $|\xi_0| \leq 3^{-1/2}$ . Then it is easy to see

$$\sum_{t^{-1/2} \leq 2^k \leq 2} J_k \lesssim \sum_{t^{-1/2} \leq 2^k \leq 2, l \geq l_0} \left| \int_{\mathbb{R}} e^{i\Phi} \widehat{P_k f}(\xi) \varphi_l^{(l_0)}(\xi - \xi_0) d\xi \right| =: \sum_{t^{-1/2} \leq 2^k \leq 2, l \geq l_0} J_{k,l},$$

where  $l_0$  is the smallest integer satisfying  $2^{l_0} \geq t^{-1/2}$  and

$$\varphi_l^{(l_0)}(\xi - \xi_0) := \begin{cases} \varphi(|\xi - \xi_0|/2^l) - \varphi(|\xi - \xi_0|/2^{l-1}), & l \geq l_0 + 1, \\ \varphi(|\xi - \xi_0|/2^{l_0}), & l = l_0 \end{cases}$$

with  $\varphi$  the smooth function given in Section 1. By this definition,  $\partial_\xi \Phi$  vanishes in the integral domain of  $J_{k,l_0}$ , and we estimate this term as

$$J_{k,l_0} \leq \|\widehat{P_k f}(\xi)\|_{L^\infty} \|\varphi_{l_0}^{(l_0)}(\xi - \xi_0)\|_{L^1} \lesssim t^{1/2} 2^{l_0} \lesssim 1.$$

For  $l \geq l_0 + 1$ , note that  $|\partial_\xi^2 \Phi| \sim t$  and

$$|\partial_\xi \Phi| = |\partial_\xi \Phi(\xi) - \partial_\xi \Phi(\xi_0)| = |\partial_\xi^2 \Phi(\xi^*)| |\xi - \xi_0| \sim t 2^l,$$

so integrating by parts in  $\xi$  as (A.5), we can obtain

$$\begin{aligned} \sum_{t^{-1/2} \leq 2^k \leq 2, l > l_0} J_{k,l} &\lesssim \sum_{l > l_0} \left( t^{-1} 2^{-2l} \|\widehat{P_k f}\|_{L^\infty} \|\varphi_l^{(l_0)}(\xi - \xi_0)\|_{L^1} \right. \\ &\quad \left. + t^{-1} 2^{-l} \|\partial_\xi \widehat{P_k f}\|_{L^2} \|\varphi_l^{(l_0)}(\xi - \xi_0)\|_{L^2} + t^{-1} 2^{-l} \|\widehat{P_k f}\|_{L^\infty} \|\partial_\xi \varphi_l^{(l_0)}(\xi - \xi_0)\|_{L^1} \right) \\ &\lesssim \sum_{l > l_0} (t^{-1} 2^{-2l} t^{1/2} 2^l + t^{-1} 2^{-l} t^{5/8} 2^{l/2} + t^{-1} 2^{-l} t^{1/2}) \lesssim 1. \end{aligned}$$

This ends the proof of the lemma.  $\square$

## Appendix B

In this appendix, we collect some analysis lemmas.

**Lemma B.1** *There holds*

$$\|\mathcal{O}[f, M]V\|_{L^2(\mathbb{R})} \lesssim \|M(\xi, \eta - \xi)\|_{L_\eta^\infty H_\xi^1} \|f\|_{L^\infty(\mathbb{R})} \|V\|_{L^2(\mathbb{R})}. \quad (\text{B.1})$$

*Proof.* Let  $\mathcal{F}_x^\xi$  denote the Fourier transform from  $x$  to  $\xi$ . Using Hölder's inequality, we can see

$$\begin{aligned} |\langle \mathcal{O}[f, M]V, W \rangle| &= \frac{1}{(2\pi)^2} \left| \int_{\mathbb{R}^2} M(\xi, \eta - \xi) \widehat{f}(\xi) \widehat{V}(\eta - \xi) \overline{\widehat{W}}(\eta) d\xi d\eta \right| \\ &= \frac{1}{(2\pi)^2} \left| \int_{\mathbb{R}} \left( \int_{\mathbb{R}} M(\xi, \eta - \xi) \mathcal{F}_x^\xi \mathcal{F}_y^\eta(f(x+y)V(y)) d\xi \right) \overline{\widehat{W}}(\eta) d\eta \right| \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathcal{F}_\xi^{-1} M(\xi, \eta - \xi)(x) \mathcal{F}_y^\eta(f(-x+y)V(y)) dx \right) \overline{\widehat{W}}(\eta) d\eta \right| \\ &\lesssim \int_{\mathbb{R}} \|\langle x \rangle \mathcal{F}_\xi^{-1} M(\xi, \eta - \xi)\|_{L_x^2} \cdot \|\langle x \rangle^{-1} \mathcal{F}_y^\eta(f(-x+y)V(y))\|_{L_x^2} \cdot |\overline{\widehat{W}}(\eta)| d\eta \\ &\lesssim \|\langle x \rangle \mathcal{F}_\xi^{-1} M(\xi, \eta - \xi)\|_{L_\eta^\infty L_x^2} \cdot \|\langle x \rangle^{-1} \mathcal{F}_y^\eta(f(-x+y)V(y))\|_{L_\eta^2 L_x^2} \cdot \|\overline{\widehat{W}}(\eta)\|_{L_\eta^2} \\ &\lesssim \|M(\xi, \eta - \xi)\|_{L_\eta^\infty H_\xi^1} \cdot \|\langle x \rangle^{-1} f(-x+y)V(y)\|_{L_y^2 L_x^2} \cdot \|W\|_{L^2}. \end{aligned}$$

Note that

$$\|\langle x \rangle^{-1} f(-x + y) V(y)\|_{L_y^2 L_x^2} \lesssim \|f\|_{L^\infty} \|V\|_{L^2},$$

then the desired estimate (B.1) follows by duality argument.  $\square$

**Lemma B.2** *Let  $m(\xi, \eta)$  be a Fourier multiplier satisfying*

$$\|m\|_{L^2(\mathbb{R}^2)} + \|\partial_\xi^2 m\|_{L^2(\mathbb{R}^2)} + \|\partial_\eta^2 m\|_{L^2(\mathbb{R}^2)} \lesssim 1, \quad (\text{B.2})$$

then for any  $p_0, p_1, p_2 \in [1, +\infty]$  with  $p_0^{-1} = p_1^{-1} + p_2^{-1}$ , we have

$$\|\mathcal{O}[f_1, m]f_2\|_{L^{p_0}(\mathbb{R})} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}. \quad (\text{B.3})$$

*Proof.* Define

$$K(x, y) := \mathcal{F}^{-1}[m(\xi, \eta)] = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x\xi + y\eta)} m(\xi, \eta) d\xi d\eta.$$

Note that

$$\mathcal{F}_{\tilde{x}}^\xi[f_1(x + \tilde{x})] = e^{ix\xi} \widehat{f}_1(\xi), \quad \mathcal{F}_{\tilde{y}}^\eta[f_2(x + \tilde{y})] = e^{ix\eta} \widehat{f}_2(\eta),$$

where  $\mathcal{F}_{\tilde{x}}^\xi$  is the Fourier transform from  $\tilde{x}$  to  $\xi$ , then by (1.12),

$$\begin{aligned} (\mathcal{O}[f_1, m]f_2)(x) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix(\xi + \eta)} m(\xi, \eta) \widehat{f}_1(\xi) \widehat{f}_2(\eta) d\xi d\eta \\ &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \mathcal{F}_{\tilde{x}}^\xi \mathcal{F}_{\tilde{y}}^\eta [K(\tilde{x}, \tilde{y})] \mathcal{F}_{\tilde{x}}^\xi \mathcal{F}_{\tilde{y}}^\eta [f_1(x + \tilde{x}) f_2(x + \tilde{y})] d\xi d\eta \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} K(\tilde{x}, \tilde{y}) f_1(x - \tilde{x}) f_2(x - \tilde{y}) d\tilde{x} d\tilde{y}, \end{aligned}$$

where we have used the identity  $\langle \widehat{F}, \widehat{G} \rangle = (2\pi)^2 \langle F, G \rangle$  ( $F, G : \mathbb{R}^2 \rightarrow \mathbb{C}$ ) in the last step. Hence, by Hölder's inequality,

$$\begin{aligned} \|\mathcal{O}[f_1, m]f_2\|_{L^{p_0}(\mathbb{R})} &\lesssim \int_{\mathbb{R}^2} |K(\tilde{x}, \tilde{y})| \cdot \|f_1(x - \tilde{x})\|_{L_x^{p_1}(\mathbb{R})} \|f_2(x - \tilde{y})\|_{L_x^{p_2}(\mathbb{R})} d\tilde{x} d\tilde{y} \\ &\lesssim \|K\|_{L^1(\mathbb{R}^2)} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})} \end{aligned} \quad (\text{B.4})$$

with  $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$ . Moreover, using (B.2), we have

$$\begin{aligned} \|K(x, y)\|_{L^1(\mathbb{R}^2)} &\leqslant \|(1 + x^2 + y^2)^{-1}\|_{L^2(\mathbb{R}^2)} \|(1 + x^2 + y^2)K(x, y)\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|K(x, y)\|_{L^2(\mathbb{R}^2)} + \|x^2 K(x, y)\|_{L^2(\mathbb{R}^2)} + \|y^2 K(x, y)\|_{L^2(\mathbb{R}^2)} \\ &\sim \|m(\xi, \eta)\|_{L^2(\mathbb{R}^2)} + \|\partial_\xi^2 m(\xi, \eta)\|_{L^2(\mathbb{R}^2)} + \|\partial_\eta^2 m(\xi, \eta)\|_{L^2(\mathbb{R}^2)} \\ &\lesssim 1. \end{aligned} \quad (\text{B.5})$$

Therefore, the desired bound (B.3) follows from (B.4) and (B.5).  $\square$

**Lemma B.3** If  $m(\eta, \sigma)$  is a Fourier multiplier with  $\eta$  and  $\sigma$  localized in the size  $2^k$  and  $2^l$ , respectively, and satisfies

$$|\partial_\eta^a \partial_\sigma^b m| \lesssim A 2^{-ak} 2^{-bl} \text{ (resp. } A)$$

for any  $a, b = 0, 1, 2$ , then we have

$$\|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^2)} \lesssim A \text{ (resp. } A 2^k 2^l).$$

*Proof.* Let  $K(x, y) := \mathcal{F}^{-1}m$ , namely,

$$K(x, y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\eta} e^{iy\sigma} m(\eta, \sigma) d\eta d\sigma.$$

We first assume

$$|\partial_\eta^a \partial_\sigma^b m| \lesssim A 2^{-ak} 2^{-bl}, \quad a, b = 0, 1, 2. \quad (\text{B.6})$$

Using the localized property of  $m$ , we see that

$$|K(x, y)| \lesssim \|m\|_{L^\infty} 2^k 2^l \leqslant A 2^k 2^l, \quad \forall (x, y) \in \mathbb{R}^2. \quad (\text{B.7})$$

On the other hand, with integration by parts, it is easy to see

$$|K(x, y)| \lesssim x^{-a} y^{-b} \|\partial_\eta^a \partial_\sigma^b m\|_{L^\infty} 2^k 2^l, \quad x \neq 0 \text{ and } y \neq 0. \quad (\text{B.8})$$

Let  $\mathbb{R}^2 = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ , where

$$\begin{aligned} \Omega_1 &= \{(x, y); |x| \leqslant \alpha, |y| \leqslant \beta\}, & \Omega_2 &= \{(x, y); |x| \leqslant \alpha, |y| \geqslant \beta\}, \\ \Omega_3 &= \{(x, y); |x| \geqslant \alpha, |y| \leqslant \beta\}, & \Omega_4 &= \{(x, y); |x| \geqslant \alpha, |y| \geqslant \beta\}. \end{aligned}$$

Then using (B.7), there holds

$$\|K(x, y)\|_{L^1(\Omega_1)} \lesssim \alpha \beta A 2^k 2^l.$$

Integrating by parts in  $\sigma$  only and using (B.6), (B.8) with  $(a, b) = (0, 2)$ , we obtain

$$\|K(x, y)\|_{L^1(\Omega_2)} \lesssim \alpha \beta^{-1} \|\partial_\sigma^2 m\|_{L^\infty} 2^k 2^l \lesssim \alpha \beta^{-1} A 2^k 2^{-l}.$$

Similarly, we can obtain

$$\|K(x, y)\|_{L^1(\Omega_3)} \lesssim \alpha^{-1} \beta \|\partial_\eta^2 m\|_{L^\infty} 2^k 2^l \lesssim \alpha^{-1} \beta A 2^{-k} 2^l.$$

Also, with integration by parts in  $\eta$  and  $\sigma$ , we have

$$\|K(x, y)\|_{L^1(\Omega_4)} \lesssim \alpha^{-1} \beta^{-1} \|\partial_\eta^2 \partial_\sigma^2 m\|_{L^\infty} 2^k 2^l \lesssim \alpha^{-1} \beta^{-1} A 2^{-k} 2^{-l}.$$

Now, we choose  $\alpha, \beta$  satisfying  $\alpha 2^k = 1$  and  $\beta 2^l = 1$ , then from the above four estimates, there holds

$$\|\mathcal{F}^{-1}m\|_{L^1(\mathbb{R}^2)} = \|K\|_{L^1(\mathbb{R}^2)} \lesssim A.$$

Next, we assume  $|\partial_\eta^a \partial_\sigma^b m| \lesssim A$  for any  $a, b = 0, 1, 2$ . In this case, applying the same argument as above with  $\alpha = \beta = 1$ , we can easily see that  $\|K\|_{L^1(\mathbb{R}^2)} \lesssim A 2^k 2^l$ . This ends the proof of the lemma.  $\square$

**Lemma B.4** *For any  $\lambda, \mu > 0$  and  $n \in \mathbb{N}$ , there holds that*

$$\int_{\mathbb{R}^2} e^{i\lambda xy} \varphi(\mu^{-1}x) \varphi(\mu^{-1}y) dx dy = 2\pi \lambda^{-1} + \lambda^{-1-n} \mu^{-2n} O(1), \quad (\text{B.9})$$

where  $\varphi$  is the smooth radial function used in the Littlewood-Paley decomposition. The implicit constant coming from the term  $O(1)$  depends only on  $n$  and  $\varphi$ .

*Proof.* We first set  $\lambda = 1$ . A direct computation gives

$$\begin{aligned} \text{LHS of (B.9)} &= \mu \int_{\mathbb{R}} \varphi(\mu^{-1}x) \widehat{\varphi}(-\mu x) dx = \int_{\mathbb{R}} \varphi(\mu^{-2}x) \widehat{\varphi}(x) dx \\ &= \int_{\mathbb{R}} \varphi(0) \widehat{\varphi}(x) dx + \int_{\mathbb{R}} [\varphi(\mu^{-2}x) - \varphi(0)] \widehat{\varphi}(x) dx \\ &= 2\pi + \int_{\mathbb{R}} [\varphi(\mu^{-2}x) - \varphi(0)] \widehat{\varphi}(x) dx, \end{aligned}$$

since  $\int_{\mathbb{R}} \widehat{\varphi}(x) dx = 2\pi \varphi(0) = 2\pi$ . Using Taylor's expansion, we have

$$\varphi(\mu^{-2}x) = \varphi(0) + \sum_{k=1}^{n-1} \frac{\varphi^{(k)}(0)}{k!} \left(\frac{x}{\mu^2}\right)^k + \frac{\varphi^{(n)}(y)}{n!} \left(\frac{x}{\mu^2}\right)^n = \varphi(0) + \frac{\varphi^{(n)}(y)}{n!} \left(\frac{x}{\mu^2}\right)^n,$$

where  $0 < |y| < \mu^{-2}|x|$ . Hence, there holds

$$\int_{\mathbb{R}} [\varphi(\mu^{-2}x) - \varphi(0)] \widehat{\varphi}(x) dx = (\mu^{2n} n!)^{-1} \int_{\mathbb{R}} x^n \widehat{\varphi}(x) \varphi^{(n)}(y) dx.$$

Combining the above equalities, we obtain

$$\int_{\mathbb{R} \times \mathbb{R}} e^{-ixy} \varphi(\mu^{-1}x) \varphi(\mu^{-1}y) dx dy = 2\pi + \mu^{-2n} O(1),$$

and by transformation  $\sqrt{\lambda}x \rightarrow x$ ,  $\sqrt{\lambda}y \rightarrow y$ , we thus get (B.9) as desired.  $\square$

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